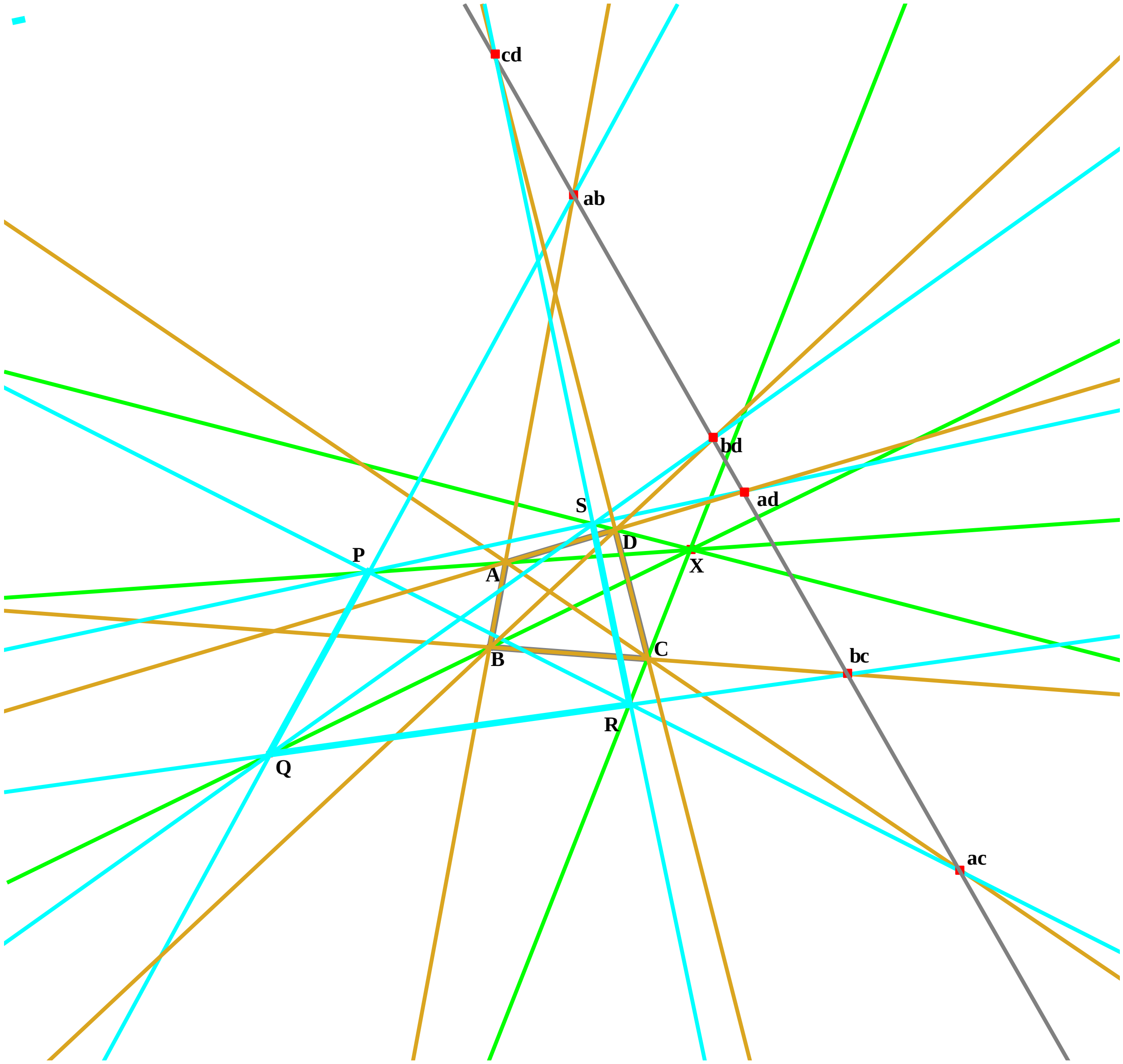
Article: CJB/238/2012

When Two Quadrilaterals are in Complete Perspective

Christopher Bradley

Abstract: If two quadrilaterals ABCD and PQRS are such that AP, BQ, CR, DS are concurrent at X then S may be moved into one and only one new position on DX so that the four Desargues' axes coincide.



ABCD and PQRS in Complete Perspective

1. Summary

If two quadrilaterals ABCD and PQRS are such that AP, BQ, CR, DS are concurrent at a point X, then there are four Desargues' axes of perspective corresponding to the four triangles ABC, ABD, ACD, BCD. It is shown in this article that S may be moved into a new position so that the four Desargues' axes coincide and that the new position of S is unique. We then say that the quadrilaterals are in *complete perspective*.

2. Positions of the vertices of the quadrilateral

We use projective co-ordinates throughout with ABC as triangle of reference and D the point with co-ordinates D(f, g, h).

We suppose that the point X of concurrence of AP, BQ, CR, DS has co-ordinates X(1, 1, 1). The there exist constants s, k, m, n such that the co-ordinates of P, Q, R, S are P(1 + s, s, s),
Q(k, 1 + k, k), R(m, m, 1 + m), S(f + n, g + n, h + n).

3. The equations of the sides and diagonals of the quadrilaterals

With the co-ordinates as specified in Section 2 the equations of the sides and diagonals of the two quadrilaterals are:

AB: $z = 0$, (3.1)
BC: $x = 0$, (3.2)
CD: $fy = gx$, (3.3)
DA: $gz = hy$, (3.4)
AC: $y = 0$, (3.5)
BD: $fz = hx$, (3.6)
PQ: $(k + s + 1)z = sx + ky$, (3.7)
QR: $(k + m + 1)x = ky + mz$, (3.8)
RS: $(g(m + 1) – hm + n)x = (f(m + 1) – hm + n)y + m(g – f)z$, (3.9)
SP: $(h – g)sx + (fs – h(s + 1) – n)y = (fs – g(s + 1) – n)z$, (3.10)
PR: $sx + mz = (m + s + 1)y$, (3.11)
QS: $k(f – h)y = (gk – h(k + 1) – n)x + (f(k + 1) – gk + n)z$. (3.12)

4. The points where corresponding sides meet

AB^PQ = ab: (– k, s, 0), (4.1)
BC^QR = bc: (0 – m, k), (4.2)
CD^RS = cd: (fm, gm, hm – n), (4.3)
DA^SP = da: (fs – n, gs, hs), (4.4)

AC^PR = ac: (– m, 0, s), (4.5)
BD^QS = bd: (fk, gk – n, hk). (4.6)

5. The four Desargues' lines of perspective and when they coincide

Line ab bc ac: sx + ky + mz = 0, (5.1)
Line ab bd da: hsx + hky = (fs + gk – n)z, (5.2)
Line ac da cd: gsx + gmz = (fs + hm – n)y, (5.3)
Line bc cd bd: fky + fmz = (gk + hm – n)x. (5.4)

These four Desargues' lines of perspective coincide if and only if

$$n = fs + gk + hm. \tag{5.5}$$

This means that there is one and only one point on DX for the point S such that the four Desargues' lines of perspective coincide.

Thus a construction for two quadrilaterals ABCD and PQRS to be in complete perspective is as follows: Let ABCD be any convex quadrilateral. Choose PQR to be any triangle such that AP, BQ, CR are concurrent at a point X. Now there is just one point on DX where S may be placed so that the four Desargues' axes coincide and then we have complete perspective.

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.