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Special Conical Hexagons

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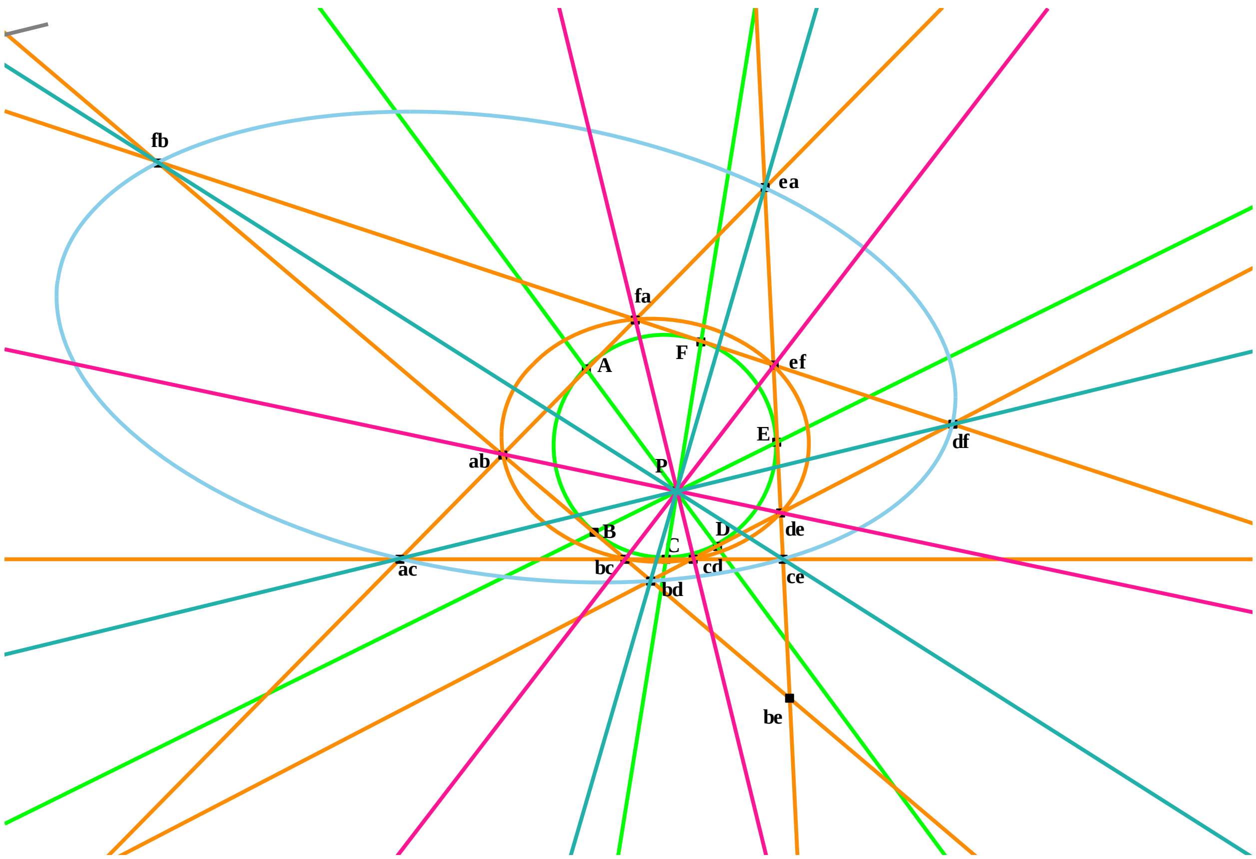
Abstract: A Hexagon ABCDEF inscribed in a Conic is said to be ‘special’ if AD, BE, CF are concurrent at a point P. When this happens hexagons formed by taking the intersections of neighbouring tangents are proved also to be special and conical with the same point P.



1. Introduction

We take the inner conical hexagon ABCDEF (without loss of generality) to lie on a circle. The tangents at A and B meet at ab and in general the tangents at X and Y ≠ X meet at xy. We prove that ab de, bc ef and cd fa all pass through P and Cabri shows the six points lie on a conic and thereby form a special conical hexagon. It is also shown that ca bd, fb ce and ea bd all pass through P and that Cabri again shows these six points also form a special conical hexagon.

2. The points

We take points on the unit circle to have parameters a, b, c, d, e, f and co-ordinates of the form

$$x = (1 - a^2)/(1 + a^2), \quad y = 2a/(1 \quad a^2). \qquad (2.1)$$

The tangent at A has equation

$$(1 - a^2x + 2ay = (1 + a^2), \qquad (2.2)$$

and similarly for other tangents with appropriate change of parameter.

Since AD, BE and FC meet at a point P then P has co-ordinates

$$x = -(a(b(d - e) + de - 1) + b(1 - de) - d + e)/ (a(b(d - e) + de + 1) - b(1 + de) + d - e),$$
$$y = 2(ad - be)/(a(b(d - e) + de + 1) - b(1 + de) + d - e), \qquad (2.3)$$

with the condition for concurrence being

$$abd - abe - acd + acf + ade - adf + bce - bcf - bde + bef + cdf - cef = 0. \qquad (2.4)$$

The tangents at A and B meet at the point ab with co-ordinates

$$x = (1 - ab)/(1 + ab),\ y = (a + b)/(1 + ab), \qquad (2.5)$$

and similarly for other points de, etc.

The equation of the line ab de is

$$(a(b(d + e) - de - 1) - b(de + 1) + d + e)x + 2(de - ab)y + a(b(d + e) - de + 1) + b(1 - de) - d - e = 0. \qquad (2.6)$$

Other lines in the figure have similar equations with appropriate changes of parameters.

It may be checked from the equations of lines such as (2.6) by virtue of Equations (2.3) and (2.4), that they all pass through P.

Cabri indicates that ab fa ef de cd bc forms a conic as do the points fb ea df ce bd ac. The equations of these conics are technically very difficult and their form extremely lengthy and it is too difficult to give these equations without printing error.

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