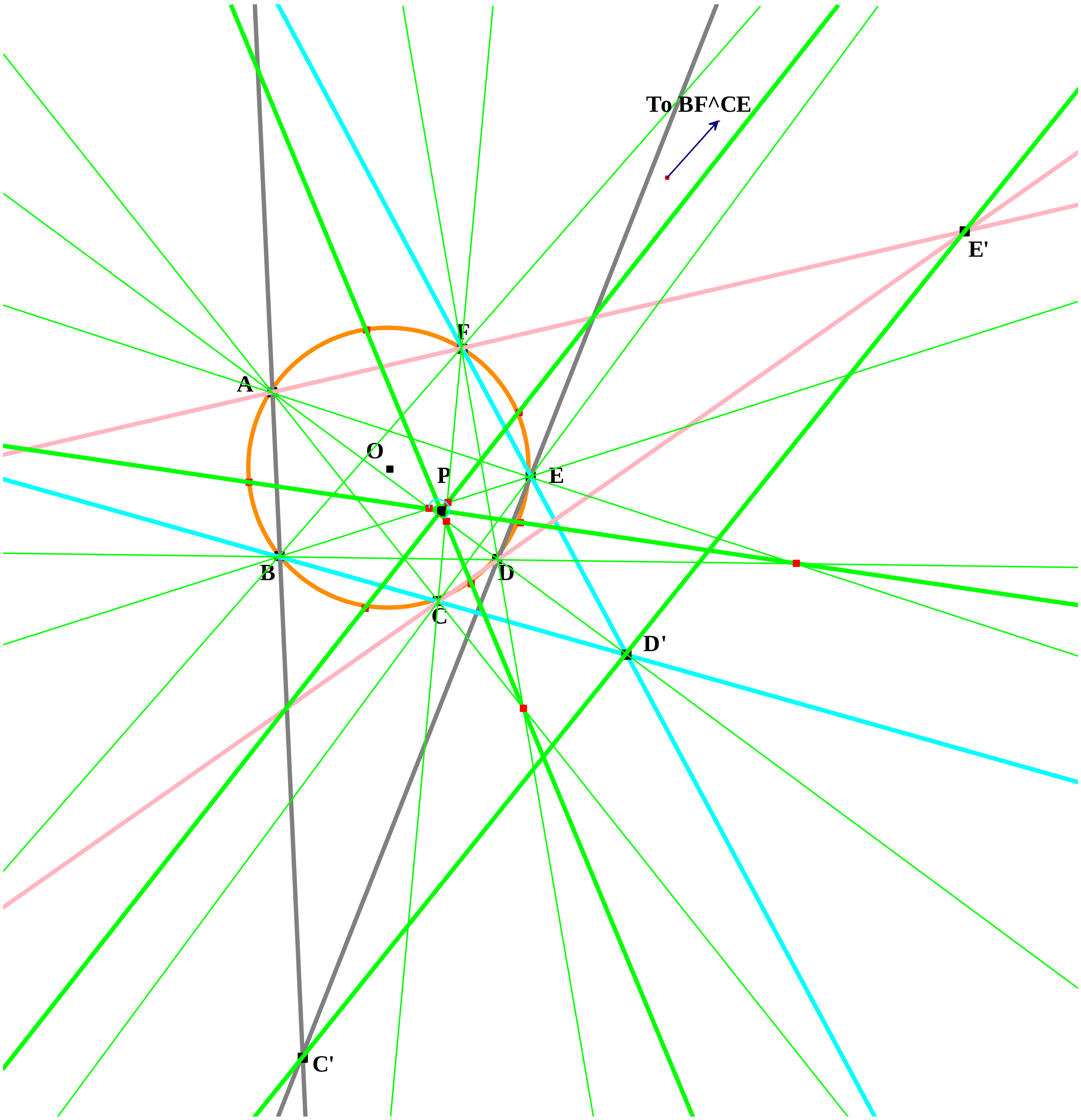
Article: CJB/224/2012

The Sixty Pascal Poles

Christopher Bradley

Abstract: When six points lie on a conic then sixty Pascal lines may be constructed. In this paper it is shown how the poles of sixty these lines may be constructed without drawing tangents. Their properties are, of course, the duals of those of the Pascal lines.



A Pascal Line and its Pole P

1. Introduction

In the figure points A, B, C, D, E, F are drawn in that order on a conic (drawn as a circle, which is sufficiently general). Points C'= AB^DE, D' = BC^EF, E' = CD^AF (that is, intersections of pairs of opposite sides are drawn) and points C'D'E' is a straight line, being the Pascal line of the ordered six points. Other orders produce altogether 60 Pascal lines. These lines have poles with respect to the defining conic and we show that the pole P of the Pascal line defined above lies on each of the three lines AD^CF AC^DF, BF^CE BE^CF, BD^AE AD^BE. The 60 poles satisfy dual properties to those of the 60 lines, and as their properties are seldom listed we mention some of these properties in the concluding section.

2. The Pascal line for the hexagon ABCDEF

This is the line C'D'E' where C' = AB^DE, D' = BC^EF, E' = CD^AF. We take the circle to have equation $x^2 + y^2 = 1$ with point A having co-ordinates $((1 - a^2)/(1 + a^2), 2a/(1 + a^2))$ and similarly for B, C, D, E, F with parameters b, c, d, e, f.

The equation of AB is well-known to be

$$(1 - ab)x + (a + b)y = (1 + ab). \tag{2.1}$$

with similar equations for other lines. The co-ordinates of C' = AB^DE may now be calculated and are (x, y), where

$$x = -(abd + abe - ade - bde - a - b + d + e)/(abd + abe - ade - bde + a + b - d - e), \tag{2.2}$$

$$y = 2(ab - de)/(abd + abe - ade - bde + a + b - d - e). \tag{2.3}$$

The co-ordinates of D' and E' follow by appropriate changes of parameters. The equation of the Pascal line is thus

$$mx + ny = 1, \tag{2.4}$$

where

$m = -(a(b(c(d - f) + ef - 1) - f(de - 1)) + bc(1 - de) + cd(ef - 1) + e(d - f))$

all divided by

$$(a(b(c(d - f) + ef + 1) - f(de + 1)) - bc(1 + de) + cd(ef + 1) - e(d - f)). \tag{2.5}$$

$n = a(b(d + e) + c(d - f) - d(e + f)) - b(c(e + f) + e(d - f)) + cf(d + e)$

all divided by

$$(a(b(c(d - f) + ef + 1) - f(de + 1)) - bc(1 + de) + cd(ef + 1) - e(d - f)). \tag{2.6}$$

3. The pole P and three lines through P

The pole P, of course, has co-ordinates (m, n), but it cannot be easily located without introducing lines through P that may be constructed. This can be done by drawing appropriate tangents to the circle from points on the Pascal line, but that is not satisfactory and indeed one can do better. We prove in fact that the lines (i) BF^CE BE^CF, (ii) BD^AE BE^ AD and (iii) AD^CF AC^DF all pass through P, so that P may be located without introducing any additional points.

In fact instead of producing 60 Pascal lines and their properties it would have been possible to produce 60 poles with consequential properties that are the duals of those properties. It is simply ***that Pascal found the line and not the point.***

We now deal with (i). The equations of BE and CF are of the form (2.1) with parameter (b, e) and (c, f) rather than (a, b). Their point of intersection BE^CF has co-ordinates that may be obtained from (2.2) and (2.3) with e, c, f replacing a, d, e. The co-ordinates of BF^CE may now be obtained from those of BE^CF by interchange of e and f. The equation of the line joining these two points is now

$$px + qy = 1, \tag{3.1}$$

where

$$p = -(b(c(e+f) - ef - 1) - c(ef+1) + e + f)/(b(c(e+f) - ef + 1) + c(1 - ef) - e - f), \tag{3.2}$$

and

$$q = 2(bc - ef)/(b(c(e+f) - ef + 1) + c(1 - ef) - e - f). \tag{3.3}$$

It may now be checked that P(m, n) lies on this line. By symmetry it now follows that P also lies on lines (ii) and (iii). Thus P may be located by using only the initial points A, B, C, D, E, F. And to each Pascal line there is a corresponding pole.

4. Some properties of the 6o poles

Corresponding to the 20 Steiner points (through any one of which 3 Pascal lines pass) we have

(i) 20 (Steiner) lines each of which has 3 poles lying on it;

Corresponding to the 60 Kirkman points (that have the property that they lie 3 at a time on lines through the 20 Steiner points) we have

(ii) 60 (Kirkman) lines such that there are 20 points of concurrences with 3 Kirkman lines through each such point and these 20 points lie one each on the 20 Steiner lines.

Similar dual correspondences exist involving analogues of the 20 Cayley lines, the 15 Plucker lines and the 15 Salmon points.

And all this occurs without mentioning Brianchon's theorem involving six tangents to a conic rather than six points on a conic as in Pascal's theorem.

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.