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Multiplication of Points using Barycentric Co-ordinates

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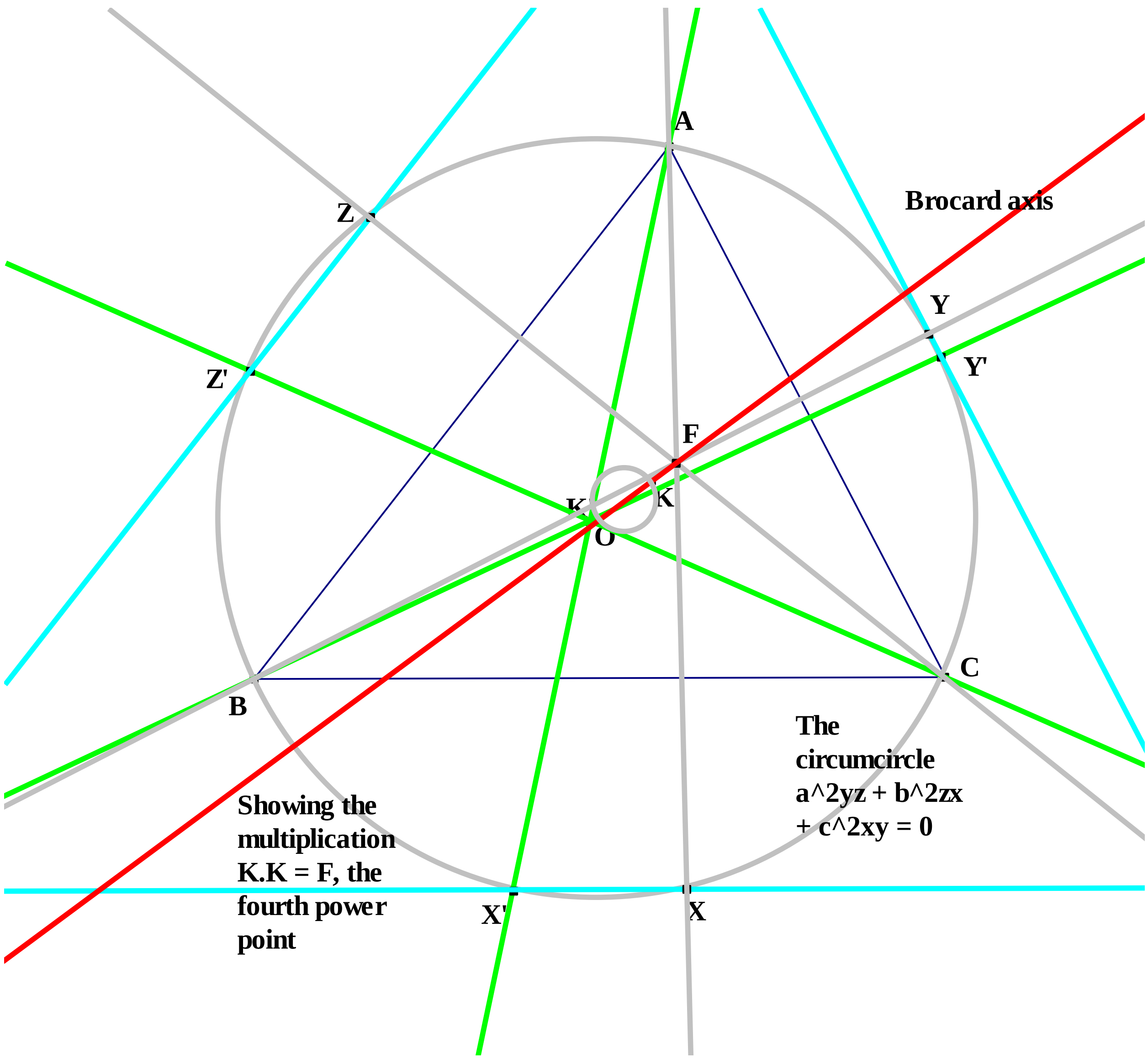
Abstract: If you have points with barycentric co-ordinates (f, g, h), (u, v, w) then the rule of multiplication is that the result has co-ordinates (fu, gv, hw). It is shown in this article how to perform this multiplication using a geometric construction . The method is as follows: First draw the circumconic fx + gy + hz = 0. Next take the isotomic conjugation of (u, v, w) to get the point (1/u, 1/v, 1/w). To conjugate this point finally use the circumconic to perform the second conjugation taking (1/u, 1/v, 1/w) to the product point (f/(1/u), g/(1/v), h/(1/v)) = (fu, gv, hw). This is evidently a commutative product and so the product may also be obtained using the conic ux + vy + wz = 0 and operating the two conjugations on the point (f, g, h).



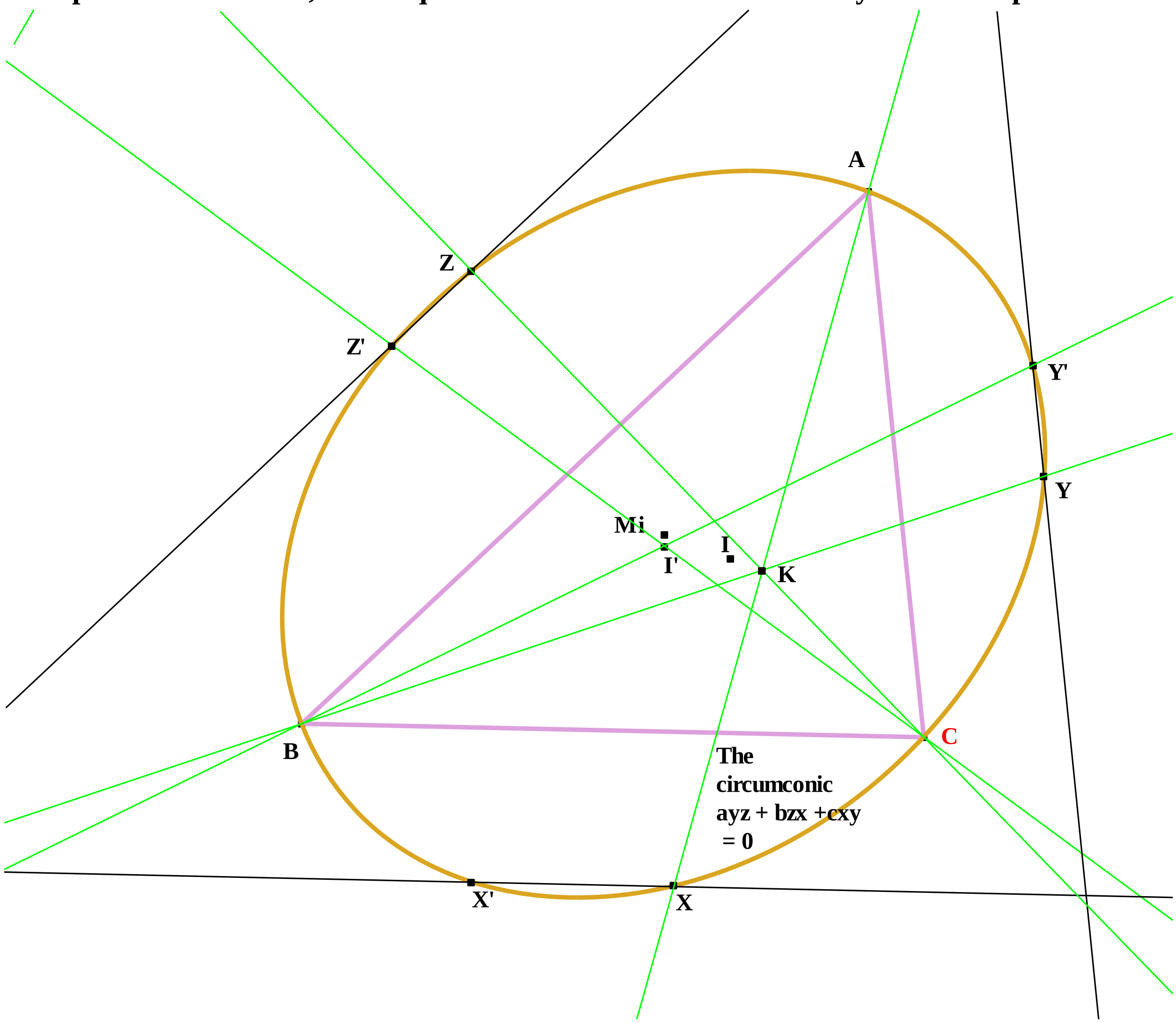
1. **The product K.K = F, the square of the symmedian point is the fourth power point**

In the above figure the product K.K = F is illustrated. It is one of the easiest products, of course (other than using the centroid G, which acts as an identity). The circumconic in this case is the circumcircle with equation

$$a^2yz + b^2zx + c^2xy = 0 \tag{1.1}$$

The first conjugation sends $K(a^2, b^2, c^2)$ to $K'(1/a^2, 1/b^2, 1/c^2)$, the isotomic conjugate of K. The second is performed as follows, and is actually the isogonal conjugation. One draws AK' to meet the circumcircle at X', then draw the line parallel to BC through X' to meet the circumcircle again at X. Using BK', CK' similarly one obtains points Y, Z also on the circumcircle. Finally AX, BY, CZ are drawn and they concur at F.

2. **The product I.I = K, the square of the incentre is the symmedian point**



Since the co-ordinates of I are I(a, b, c) the circumconic has equation

$$ayz + bzx + cxy = 0. \quad (2.1)$$

The centre of this conic is the Mittenpunkt with co-ordinates

$$Mi(a(b + c – a). b(c + a – b), c(a + b – c)). \quad (2.2)$$

The isotomic conjugation of I is I'(1/a, 1/b, 1/c) the lines AI', BI', CI' are now drawn to meet the circumconic in points X', Y, Z'. A line parallel to BC is drawn through X' to meet the circumconic at X. Points Y, Z are similarly obtained. Finally AX, BY, CZ are drawn and they concur at the symmedian point I.I = K. See the Figure on page 2.

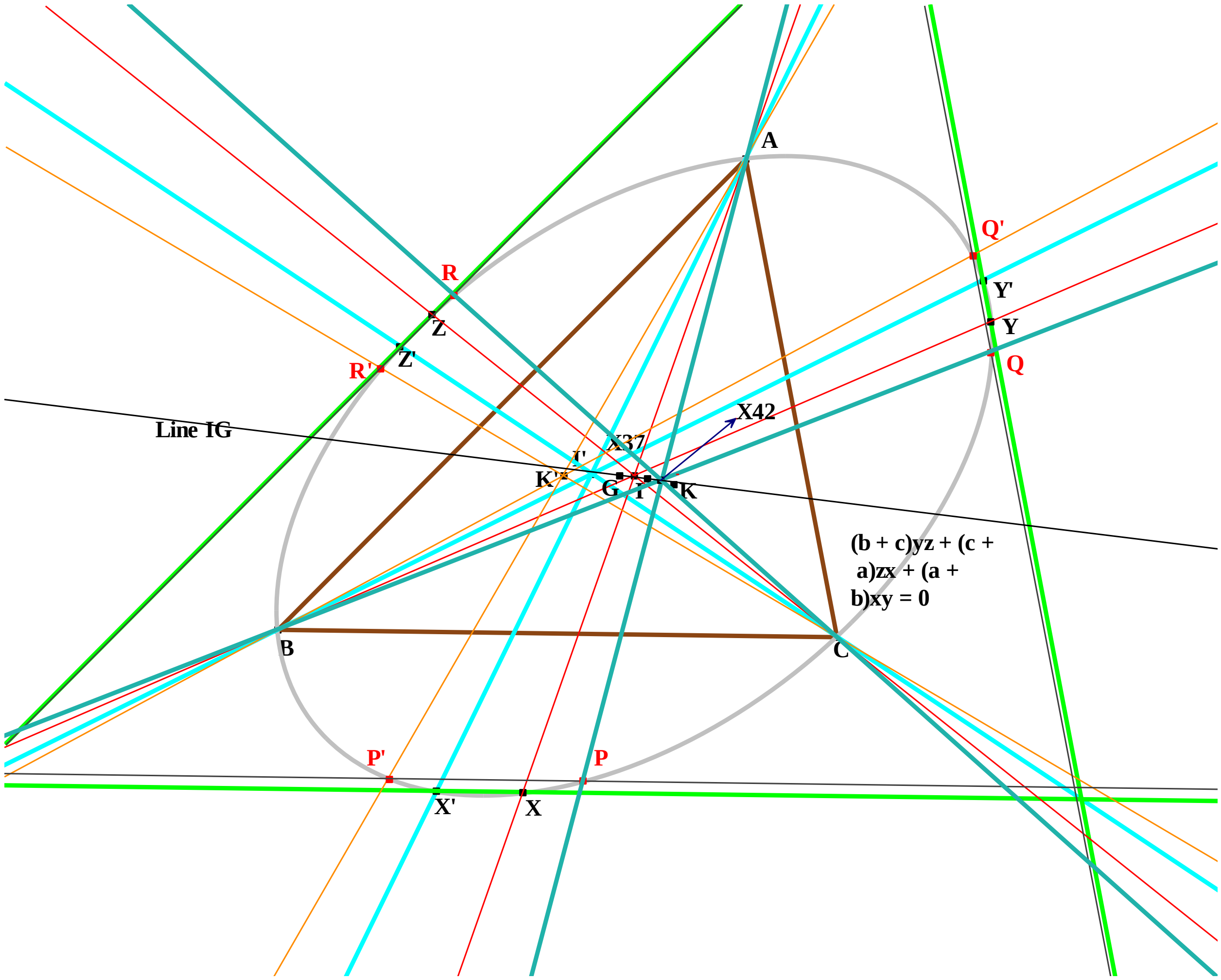
3. Two more products

In this Section we obtain two more products using the Spieker centre, for which the conic has equation

$$(b + c)yz + (c + a)zx + (a + b)xy = 0. \quad (3.1)$$

The centre of this conic is the point X_{37}, the crosspoint of the incentre and the centroid. See the figure on the next page. Using I we have I.Sp = X_{37}. This is meant to be a joke showing that to obtain X_{37} by this method you have to know where it is already. You may check that the isotomic conjugate of I, the point I', has conjugate X_{37} = (a(b + c), b(c + a), c(a + b)) = I.Sp.

When the symmedian point K is used instead of I the result is K.Sp = X_{42} the crosspoint of the incentre and symmedian point X_{42} ($a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$). X_{42} lies on the line IG, where G is the centroid. This is also illustrated in the figure following.



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