A CYCLIC QUADRILATERAL AND ITS MIDPOINT CIRCLES

CHRISTOPHER BRADLEY

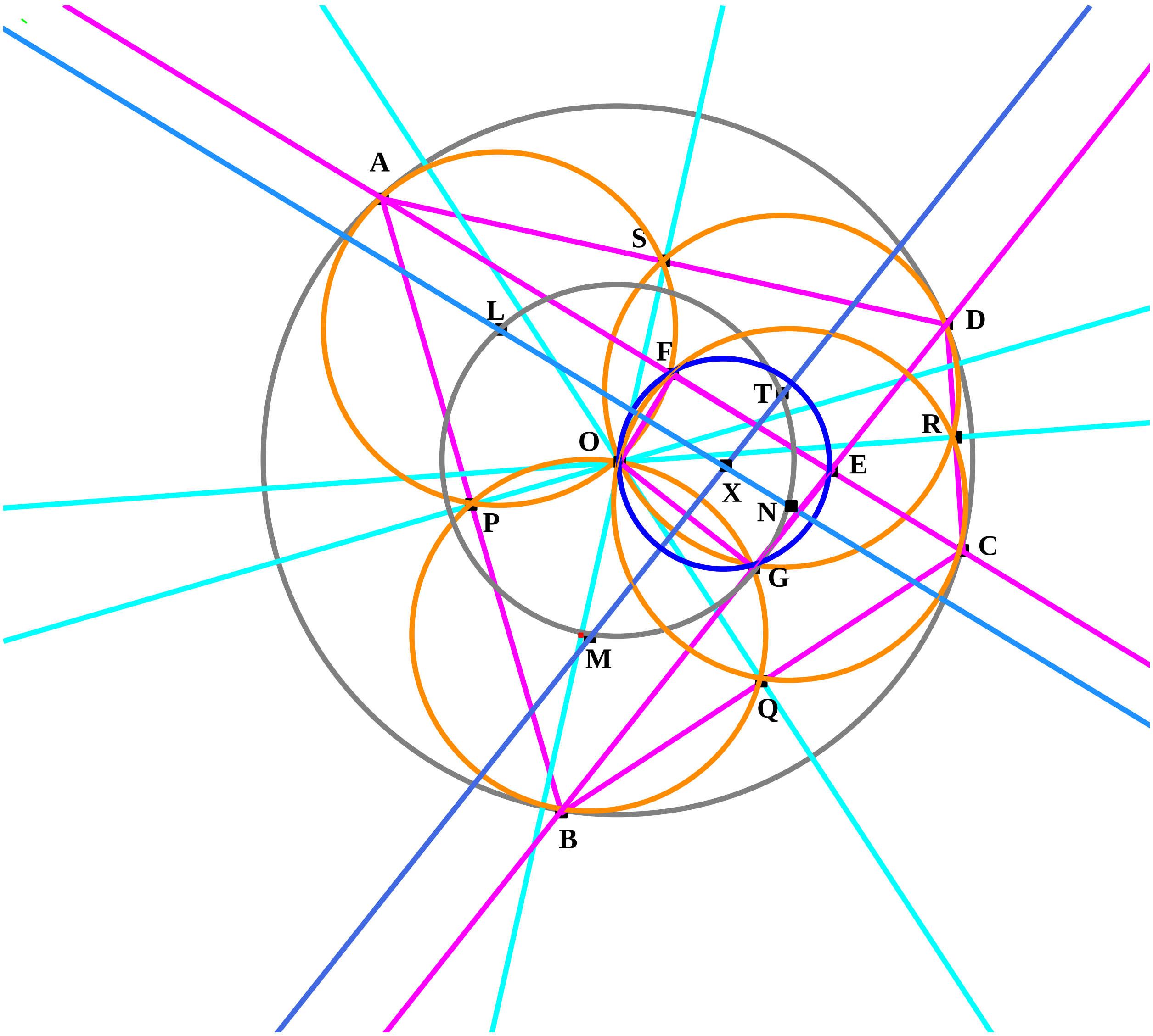

FIGURE 1

ABSTRACT:

A CYCLIC QUADRILATERAL ITS MIDPOINTS AND THE SIX MIDPOINT CIRCLES

1. Introduction

Let ABCD be a cyclic quadrilateral, P, Q, R, S the midpoints of the sides AB, BC, CD, DA respectively, O the centre of circle ABCD and E the intersection of the diagonals AC and BD. It is shown that APOS, BQOP, CROQ, DSOR are circles, having the same radius and their centres L, M, N, T are concyclic. Suppose that circle APOS meets circle CROQ at F and circle BQOP meets circle DSOR at G, then it is shown that O, E, F, G are concyclic. If the centre of circle OEFG is denoted by X then it is shown that MXT is a line parallel to BD and NXL is a line parallel to AC.

2. The points P, Q, R, S and the circles APOS, BQOP, CROQ, DSOR

We take the circle ABCD to have equation $x^2 + y^2 = 1$ and A to have co-ordinates (x, y),where

$$x = (1 – a^2)/(1 + a^2) \text{ and } y = 2a/(1 + a^2). \tag{2.1}$$

Points B, C, D have similar co-ordinates but with b, c, d instead of a. The midpoints of AO, BO, CO, DO are denoted by L, M, N, T respectively so that they have co-ordinates that are ½ of those of A, B, C, D.

The equation of the circle APOS centre L passing through O and A is

$$(1 + a^2)(x^2 + y^2) – (1 – a^2)x – 2ay = 0, \tag{2.2}$$

and this meets AB with equation

$$(1 – ab)x + (a + b)y = (1 + ab), \tag{2.3}$$

at the point P with co-ordinates (x, y), where

$$x = (1 – a^2b^2)/((1 + a^2)(1 + b^2)),\ y = (a + b)(1 + ab)/((1 + a^2)(1 + b^2)). \tag{2.4}$$

Circles BQOP, CROQ, DSOR and points Q, R, S have similar equations an co-ordinates with appropriate changes of parameters.

3. Points E, F, G, the circle OEFG and its centre X

Circles APOS and CROQ meet at the point F with co-ordinates (x, y), where

$$x = (1 – a^2c^2/((1 + a^2)(1 + c^2)),\ y = (1 + ac)(a + c)/((1 + a^2)(1 + c^2)). \tag{3.1}$$

Circles BQOP and DSOR meet at the point G with co-ordinates (x, y), where

$$x = (1 – b^2d^2)/((1 + b^2)(1 + d^2)),\ y = (1 + bd)(b + d)/((1 + b^2)(1 + d^2)). \tag{3.2}$$

We can now obtain the equation of the circle OFG which is

$$(abc + acd - abd - bcd + a - b + c - d)(x^2 + y^2) + (abc - abd + acd - bcd - a + b - c + d)x - 2(ac - bd)y = 0. \qquad (3.3)$$

And it may be checked that this circle also passes through E = AC^BD with co-ordinates (x, y), where

$$x = -(1/k)(abc - abd + acd - bcd - a + b - c + d),\ y = (1/k)(2(ac - bd))$$

and

$$k = (abc - abd + acd - bcd + a - b + c - d). \qquad (3.3)$$

The centre X of this circle therefore has co-ordinates (x, y), where

$$x = -(1/(2k))(abc - abd + acd - bcd - a + b - c + d),$$
$$y = -(1/k)(bd - ac),\ k = (abc - abd + acd - bcd + a - b + c - d). \qquad (3,4)$$

It may now be checked that the line MT passes though X and is parallel to BC and also that NL passes through X and is parallel to AC.

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.

.