Article 21

Significant Points on Circles Centre the Circumcentre

Christopher J Bradley

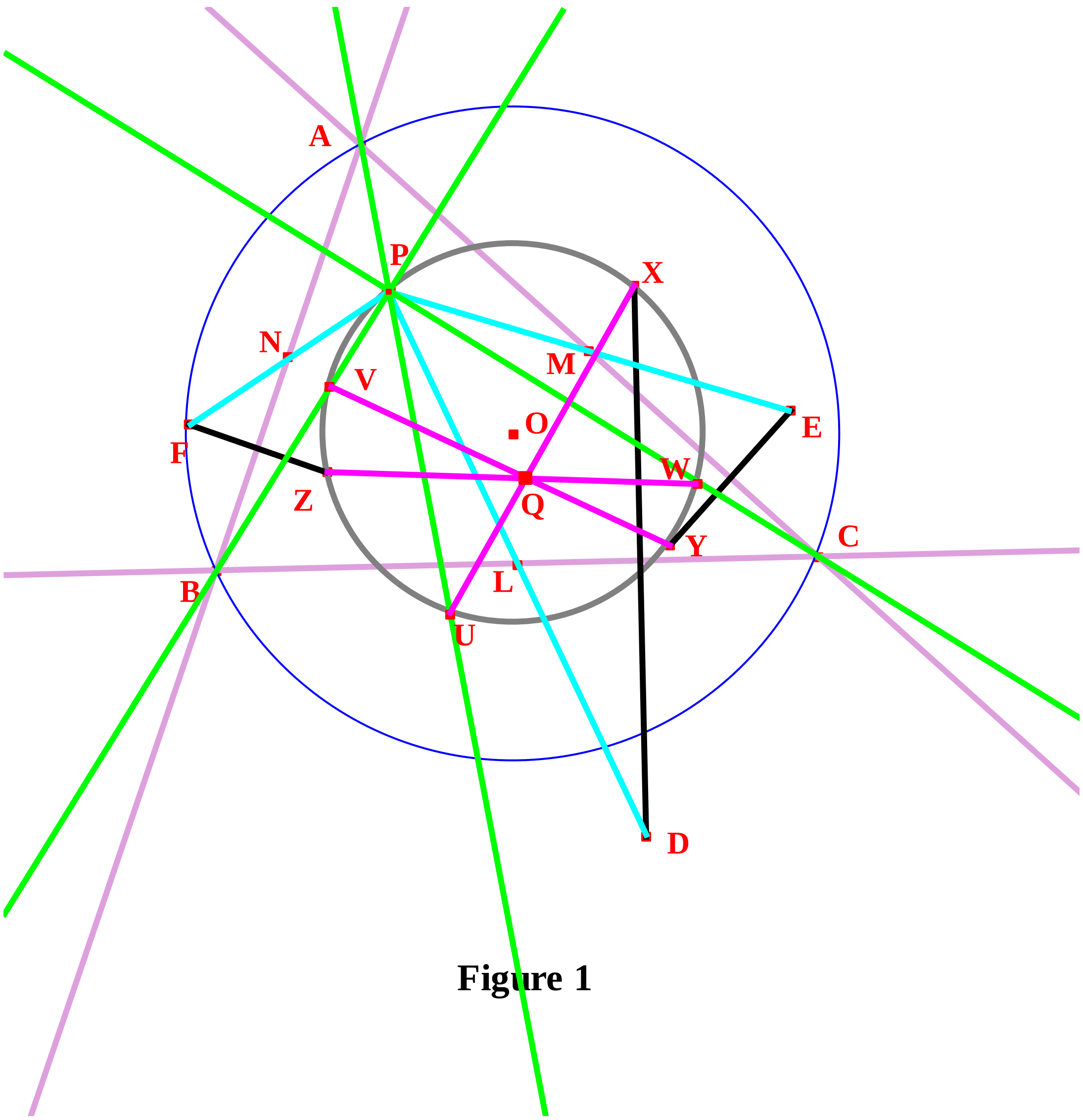

Figure 1

1. Introduction

Given a triangle ABC with circumcentre O and a point P not on its sides or their extensions and not on the circumcircle, it is shown that one may construct on the circle centre O and radius OP six significant points. The construction of U, V, W are straightforward enough; they are the second intersections of AP, BP, CP with the circle. The other three points X, Y, Z arise in a somewhat elaborate fashion. The point X, for example, arises as follows: Join P to the midpoint L of BC and extend it to D, where L is the midpoint of PD. Then reflect D in the side BC to produce X. Points Y, Z follow similarly using the midpoints M, N of the sides CA, AB respectively. Alternatively, and more easily, X is the reflection of P in the perpendicular bisector of BC. Two very interesting properties arise. First, the lines UX, VY, WZ are concurrent at a point Q. And secondly, triangle XYZ is inversely similar to triangle ABC. We prove the second result by establishing two lines of inverse similarity passing through O, with the property that ABC may be mapped into XYZ by using one of these lines and O as a centre of inverse spiral symmetry. The first property means that the inverse image T of Q under this transformation must be such that AD', BE', CF' are concurrent at T, where D', E', F' are the inverse images of U, V, W respectively. As is well known, the point T may now be used to manufacture a circle passing through the orthocentre H, known as a Hagge circle, see [1, 2]. The Hagge circle is also obtained by an inverse similarity of triangle ABC with T the centre of inverse similarity, and it follows from our analysis that the circle through P and the points on it are related to those on the Hagge circle by a direct congruence, which is just a rotation of 180^{o} about a point midway between the centres. We do not repeat any of the work on the derivation of Hagge circles in this paper, but it is our purpose to point out the connections that exist.

The procedure may, of course, be reversed. Start with a Hagge circle and rotate by 180^{o} in a way that will be explained and you end up with a circle centre O passing through the isogonal conjugate of the point that is used to construct the Hagge circle.

The article concludes with a conjecture (verified by *CABRI*, but without algebraic or geometrical proof) that other direct congruences not only map the Hagge circle and its points on to other circles with similar properties (which must obviously be the case), but the new circles manufactured have additional properties that make them interesting and significant. We believe this removes the mystique of why circles through the orthocentre are somehow thought to be special. They are no more special than any other circle once the appropriate conjugation and direct congruence are identified.

In this article we produce an analysis to prove the assertions made about our construction, using Cartesian co-ordinates. Care is needed in following the text to refer to the appropriate figure, as changes of notation from one section to the next are inevitable, so many points being involved.

2. The six points X, Y, Z, U, V, W

Let the circumcircle have equation $x^2 + y^2 = 1$, with centre O(0, 0) and suppose P has co-ordinates (p, 0). For the co-ordinates of A we use the familiar expressions $(2a/(1 + a^2), (1 – a^2)/(1$

$+ a^2$)), and similar for B and C with parameters b and c respectively. In this section and the next the notation is that of Figures 1 and 2.

L, the midpoint of BC, has co-ordinates (k(b + c), k(1 – bc)), where

$$k = (1 + bc)/\{(1 + b^2)(1 + c^2)\}. \quad (2.1)$$

Since L is the midpoint of PD we may obtain the co-ordinates of D as twice those of L minus those of P, the result being (x, y), where

$$x = [2\{(1 + bc)(b + c)\} - p(1 + b^2)(1 + c^2)]/\{(1 + b^2)(1 + c^2)\}, \quad (2.2)$$

$$y = [2(1 + bc)(1 - bc)]/ \{(1 + b^2)(1 + c^2)\}, \quad (2.3)$$

The equation of BC is

$$(b + c)x + (1 - bc)y = (1 + bc). \quad (2.4)$$

After some algebra the equation of the line perpendicular to BC passing through D is

$$(1 - bc)x - (b + c)y + p(1 - bc) = 0. \quad (2.5)$$

These lines meet at the midpoint of DX whose co-ordinates are (x, y), where

$$x = (1/s)(- b^2c^2p + (1 + bc)(b + c) + 2bcp - p)), \quad (2.6)$$

$$y = (1/s)((1 - bc)(bc + (b + c)p + 1)), \quad (2.7)$$

where $s = (1 + b^2)(1 + c^2)$. The co-ordinates of X follow and are found to be (x, y), where

$$x = (p/s)(- b^2c^2 + b^2 + c^2 + 4bc - 1), \quad (2.8)$$

$$y = (2p/s)(b + c)(1 - bc). \quad (2.9)$$

The co-ordinates of Y, Z may be written by cyclic change of a, b, c. These co-ordinates may now be substituted into the general equation of a circle, $x^2 + y^2 + 2gx + 2fy + t = 0$, to provide three equations for f, g, t, which, with help from *DERIVE*, gives f = g = 0 and $t = - p^2$, as required for the circle XYZ to have centre O and to pass through P.

The equation of AP is

$$(1 - a^2)x + ((1 + a^2)p - 2a)y - (1 - a^2)p = 0. \quad (2.10)$$

This meets the circle $x^2 + y^2 = p^2$ again at the point U with co-ordinates (x, y), where

$$x = (1/q)(p\{a^4(1 - p^2) + 4a^3p - 2a^2(p^2 + 3) + 4ap - p^2 + 1\}), \quad (2.11)$$

$$y = (2/q)(p(1 - a^2)(a^2p - 2a + p)), \quad (2.12)$$

and

$$q = a^4(1 + p^2) - 4a^3p + 2a^2(1 + p^2) - 4ap + p^2 + 1. \quad (2.13)$$

The co-ordinates of V and W may be written by replacing a by b and c respectively.

3. The point Q and the indirect similarity

We now show that the lines UX, VY, WZ are concurrent at a point Q. The equation of the line UX turns out to be

$$[a^2\{b(cp - 1) - c - p\} + 2a(1 - bc) + b(1 + cp) + c - p]x + [a^2\{b(c + p) + cp - 1\} - 2a(b + c) + b(p - c) + cp + 1]y + [a^2\{b(cp + 1) + c - p\} + 2a(1 - bc) + b(cp - 1) - c - p]p = 0. \qquad (3.1)$$

The equations of VY, WZ may be written down by cyclic change of a, b, c.

These lines are concurrent at the point Q with co-ordinates (x, y), where

$$x = (1/r)\{p(a^2(b^2(c^2(p^2 - 3) + p^2 + 1) + 4b(c - p) + c^2(p^2 + 1) - 4cp + p^2 + 1) + 4a(b^2(c - p) + b(c^2 - 2cp + 1) - c(cp - 1)) + b^2(c^2(p^2 + 1) - 4cp + p^2 + 1) + 4bc(1 - cp) + c^2(p^2 + 1) + p^2 - 3)\}, \qquad (3.2)$$

$$y = (2/r)\{p(a^2(b^2(c^2p - 2c + p) + 2bc(p - c) + p(c^2 - 1)) - 2a(b^2c(c - p) + bp(1 - c^2) + cp - 1) + b^2p(c^2 - 1) + 2b(1 - cp) - c^2p + 2c - p)\}, \qquad (3.3)$$

and

$$r = (1 - p^2)(1 + a^2)(1 + b^2)(1 + c^2). \qquad (3.4)$$

Two triangles ABC and XYZ are similar when their angles are equal and this is so if their corresponding sides are in fixed ratio, that ratio being an enlargement (or reduction) about a given point. When they are indirectly similar it is always the case that there are two axes perpendicular to one another and passing through a fixed point such that when ABC is reflected in one of these axes through the fixed point and enlarged (or reduced) by a fixed amount through the fixed point, it is mapped on to triangle XYZ. The axes are called the *double lines of inverse similarity* and the fixed point is called the *centre of inverse similarity*. Either line of symmetry may be used, but in one case the enlargement factor is positive, and in the other case it is negative, implying that a rotation of 180^o is also involved.

What we prove now is that the triangles ABC and XYZ are indirectly similar, and we do this by establishing the double lines of inverse similarity and the centre of inverse similarity. The latter is, in fact, the circumcentre O. The axes are shown in Figure 2 as the lines m and n passing through O. In the figure the axis labelled m is used. The reflection through m takes triangle ABC into triangle A'B'C' and then there is a reduction (since P is inside ABC in the case we have drawn) by a factor OX/OA' taking A' to X. Since the line m turns out to be symmetric in a, b, c and OX/OA' = p, this shows that B' is taken to Y and C' to Z, thereby establishing the similarity.

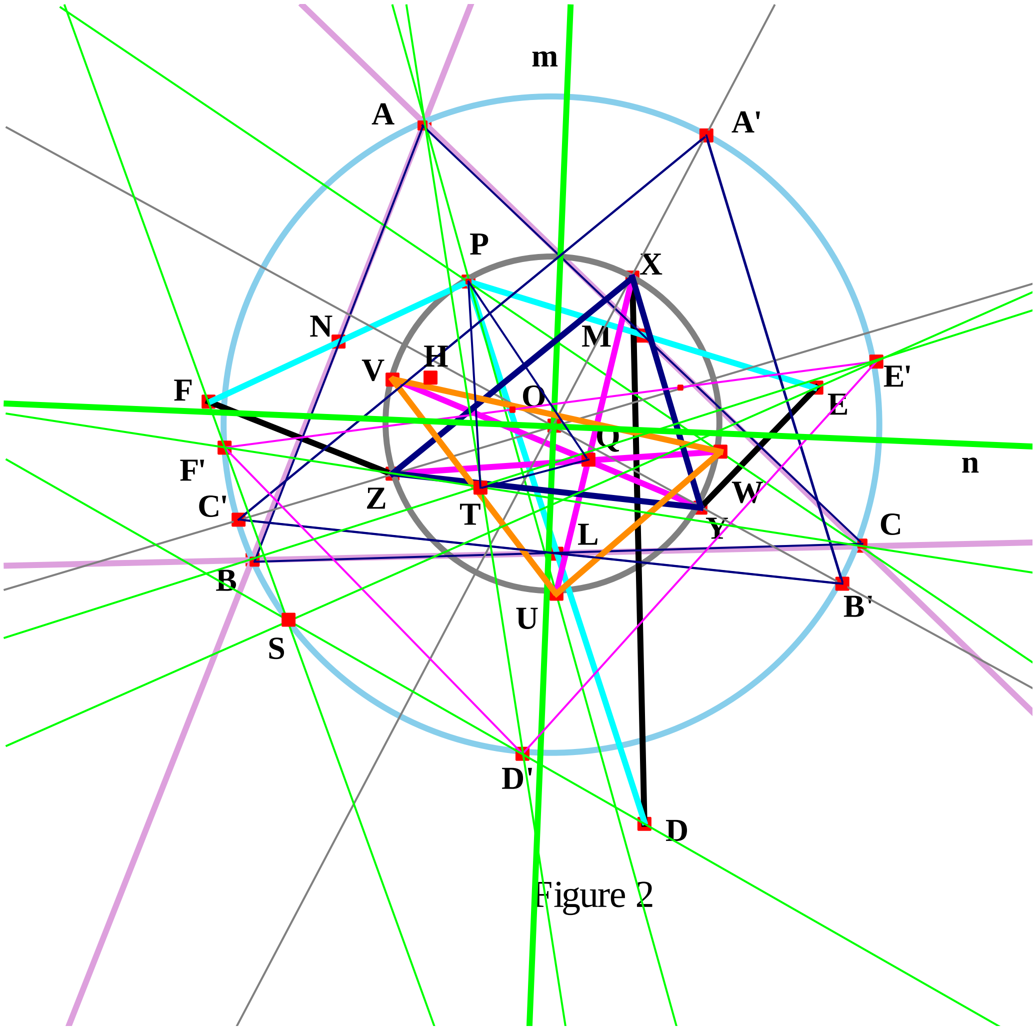

Figure 2

Suppose the equation of one of the axes of inverse symmetry passes through O and has equation y = mx. The equation of the line through A perpendicular to this line is

$$(1 + a^2)x + m(1 + a^2)y + a^2m - 2a - m = 0. \qquad (3.6)$$

These lines meet at the point with co-ordinates (x, y) where

$$x = (2a + m - a^2m)/\{(1 + a^2)(1 + m^2)\}, \qquad (3.7)$$

$$y = m(2a + m - a^2m)/\{(1 + a^2)(1 + m^2)\}. \qquad (3.8)$$

The co-ordinates of A', the reflection of A in y = mx, can now be obtained and are (x, y), where

$$x = 2(m + a)(1 - am)/\{(1 + a^2)(1 + m^2)\}, \qquad (3.9)$$

$$y = ((a - 1) + m(a + 1))((a + 1) - m(a - 1))/\{(1 + a^2)(1 + m^2)\}. \qquad (3.10)$$

We now determine the possible values of m if A', X, O are collinear. The condition for this is $x_1y_2 = x_2y_1$, where (x_1, y_1) are the co-ordinates of A' and (x_2, y_2) are the co-ordinates of X. This results in a quadratic equation for m whose solutions are, say, m and n, where mn = – 1 and

$$m = (abc + bc + ca + ab – a – b – c – 1)/(abc – bc – ca – ab – a – b – c + 1). \quad (3.11)$$

The enlargement (reduction factor) is obviously p (supposed positive without loss of generality). As the values of m and the enlargement factor are symmetric in terms of a, b, c and independent of them respectively, the proof of the indirect similarity is complete.

4. Consequences and the connection with a circle through H of radius p

Now it is well known that a Hagge circle and its key points arise from an indirect similarity of the circumcircle, A, B, C and three other key points on the circumcircle, so it is a challenge to discover a direct similarity between the circle centre O through P, that we have so far been involved with, and a Hagge circle. This proves to be possible and depends on the remarkable properties of the point T defined to be the inverse image of Q in the indirect similarity described in Section 3. See Figure 3 for an illustration of what follows.

Since T is the inverse image of Q, which we know is the point of concurrence of UX, VY, WZ, it follows that if we draw AT, BT, CT to meet the circumcircle at D', E', F' respectively, then not only is ABC indirectly similar to XYZ, but triangle D'E'F' is similar to triangle UVW. If therefore we create the Hagge circle generated by T, by reflecting D', E', F' in BC, CA, AB respectively to get points U', V', W', then triangle U'V'W' is both similar to triangle D'E'F', but then also to triangle UVW. Furthermore from the theory of Hagge circles it follows that if AH, BH, CH meet circle U'V'W' at X', Y', Z' respectively, then triangle X'Y'Z' is similar to triangle ABC and hence to triangle XYZ. Also from the theory of Hagge circles U'X', V'Y', W'Z' meet at T. This is part of what is needed to show that circle U'V'W'X'Y'Z' is directly similar to circle UVWXYZ.

The second property of T is that it is the isogonal conjugate of P. This is best proved by drawing AT and AP to meet the circumcircle at a pair of points whose displacement vector is parallel to **BC** and similarly for the other vertices. We are now very much in business, because (again quoting the theory of Hagge circles) the figure POO'H must be a parallelogram, where O' is the centre of the Hagge circle. All is now clear, because the direct similarity between U'V'W'X'Y'Z' and UVWXYZ must be a 180^{o} rotation about the midpoint R of OO'. The conclusive third property of T is that it is not only the inverse image of Q, but that R is the midpoint of QT.

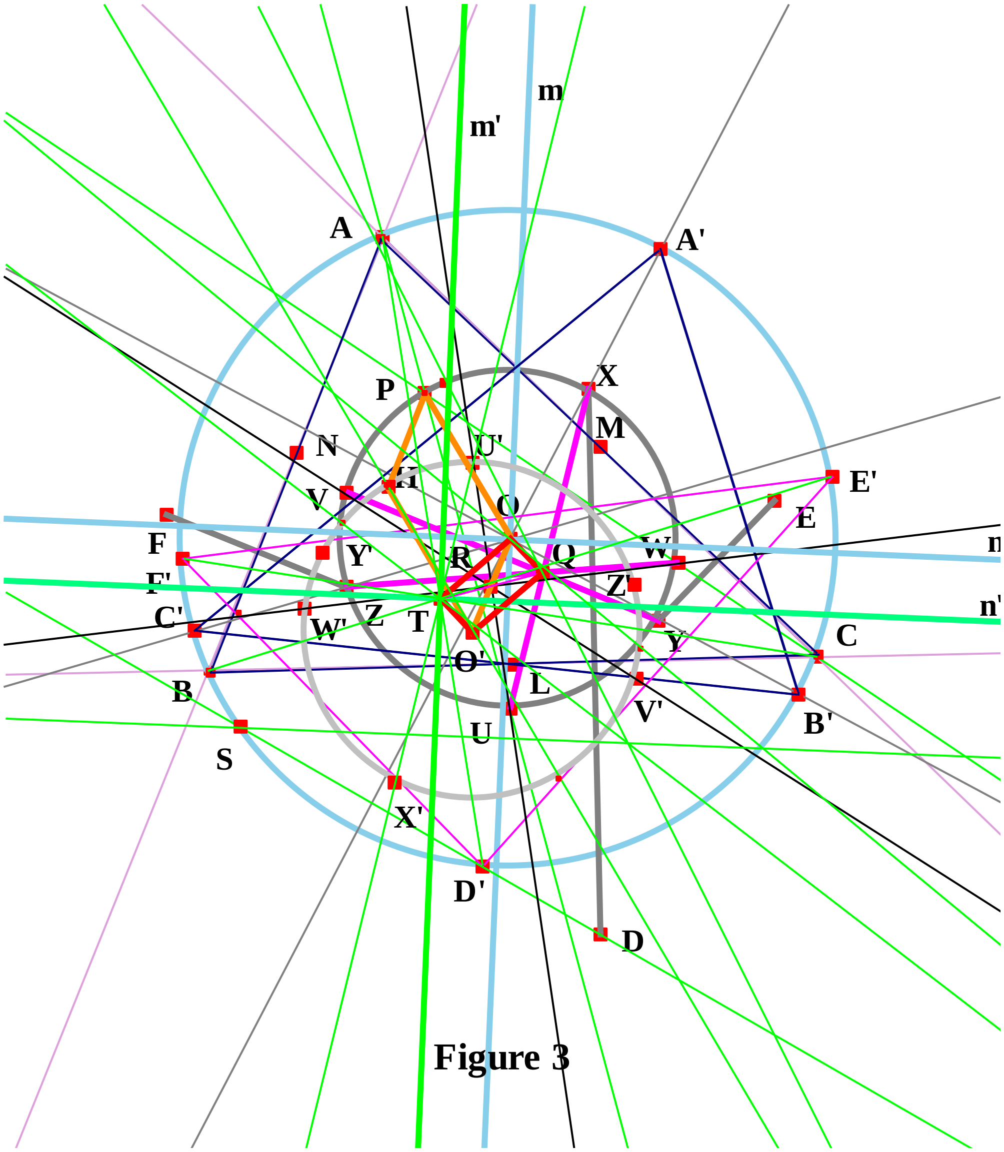

Figure 3

The whole argument may be reversed. Create a Hagge circle, centre O', from a point T, perform an 180^{o} rotation about the midpoint of OO' and you get a circle centre O of the same radius, and the remarkable thing is that this circle passes through P = Tg, the isogonal conjugate of T and, of course, TgOO'H is a parallelogram. It is also the case that R is the midpoint of UU', VV', WW', XX', YY', and ZZ'.

Further features in Figure 3 are S, the point on the circumcircle where DD', EE', FF' appear to meet, the double lines of inverse symmetry m', n' of the indirect similarity between ABC and X'Y'Z', which are parallel to the axes m, n respectively. Indeed if one reflects ABC in n' to obtain triangle A"B"C" it will be found that A"X', B"Y', C"Z' all pass through T. And finally the figure illustrates the fact that OQO'T is also a parallelogram.

What is needed in conclusion is a proof, which we now provide that the three properties leading to the position of T result in the same values for its co-ordinates. On the way we also find the co-ordinates of O'. The dilemma facing author and reader is that the algebra involved becomes technically very involved, even more so than what has preceded; so anyone checking the details will need an algebra computer package, such as *DERIVE*, which is the one we used.

In logical order we first find the co-ordinates of the point T, when defined as the pre-image of the point Q in the indirect similarity. The co-ordinates of Q are given in Equations (3.3) and (3.4), but as these are very lengthy expressions, for the time being we call them (e, f). We want the reflection of (e, f) in the line y = mx, where m is given by Equation (3.11). The line perpendicular to y = mx through (e, f) has equation x + my = e + mf. This meets y = mx at the point $((e + mf)/(1 + m^2), m(e + mf)/(1 + m^2))$. The reflection of Q has co-ordinates twice these minus those of Q. After dividing by p and inserting the values of e, f and m, this gives the following expressions for the co-ordinates of T, which are (x, y) where

$$x = (2/r)(a^2(b^2(c^2p + c(1 - p^2) - p) - b(c^2(p^2 - 1) + 2cp - p^2 - 1) - c^2p + c(p^2 + 1) - p) - a(b^2(c^2(p^2 - 1) + 2cp - p^2 - 1) + 2bp(c^2 - 2cp + 1) - c^2(p^2 + 1) + 2cp + p^2 - 1) - b^2(c^2p - c(p^2 + 1) + p) + b(c^2(p^2 + 1) - 2cp - p^2 + 1) - c^2p + c(1 - p^2) + p), \qquad (4.1)$$

$$y = (1/r)(a^2(b^2(c^2(p^2 - 3) + 4cp - p^2 - 1) + 4bcp(c - p) - (c^2 - 1)(p^2 + 1)) + 4ap(b^2c(c - p) + bp(1 - c^2) + cp - 1) + b^2(1 - c^2)(p^2 + 1) + 4bp(cp - 1) + c^2(p^2 +1) - 4cp - p^2 + 3), \qquad (4.2)$$

and where r is given by Equation (3.4).

Next we work out the isogonal conjugate of P with respect to triangle ABC and show that it coincides with T. The line AP has equation

$$(a^2 - 1)x - (a^2p - 2a + p)y - p(a^2 - 1) = 0. \qquad (4.3)$$

This meets the circumcircle at a point D'' with co-ordinates

$$(2(a^2p - a(p^2 + 1) + p), (1 - p^2)(a^2 - 1)) / (a^2 (p^2 + 1) - 4ap + p^2 + 1). \qquad (4.4)$$

The line through D'' parallel to BC meets the circumcircle again at the point D'. When the co-ordinates of A, T and D' are entered into a determinant with 1 in the last column of each row, the value of this determinant is zero. Therefore the three points are collinear. Cyclic change shows that B, T, E' and C, T, F' are also collinear and hence T is the isogonal conjugate of P.

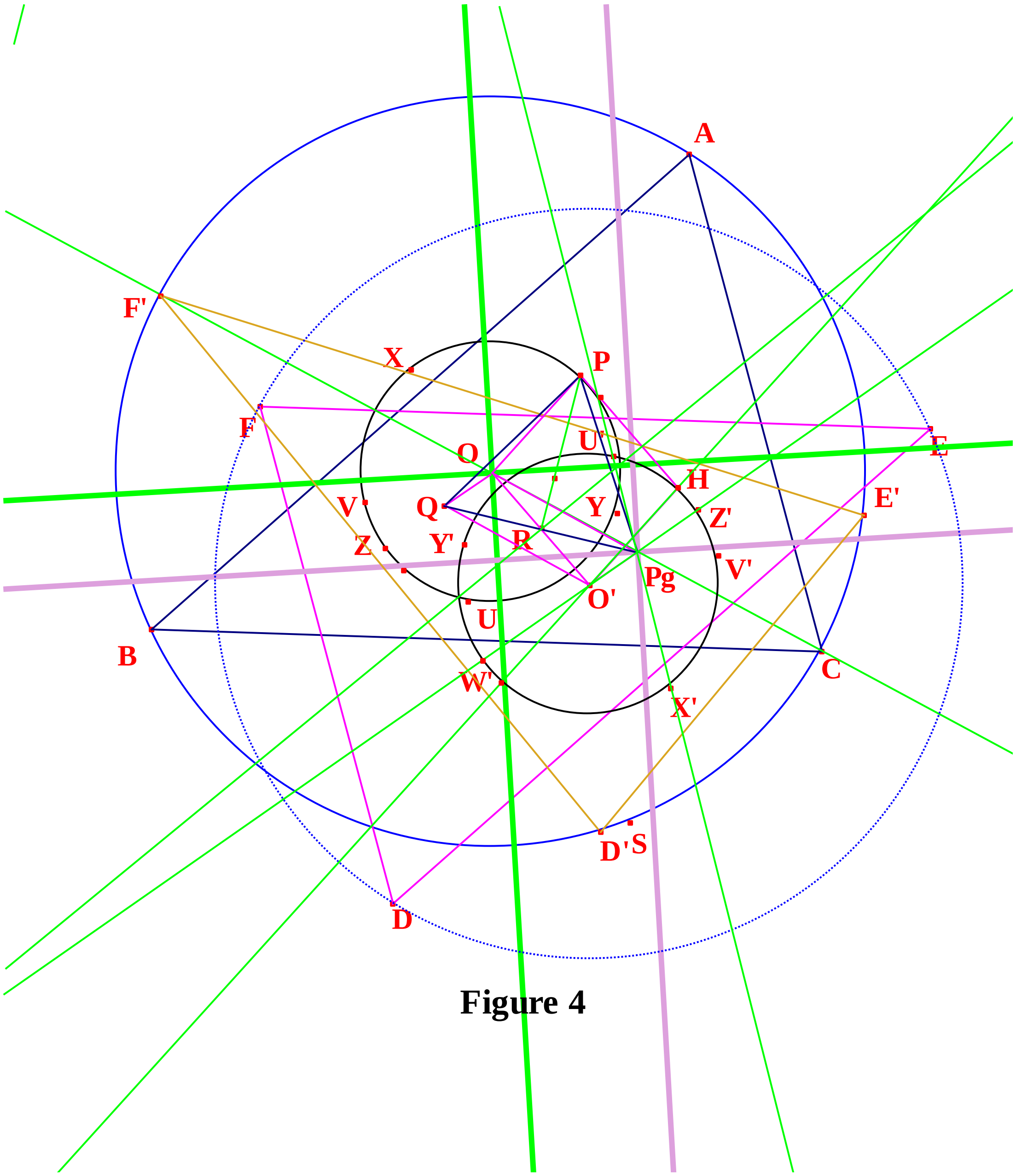

Figure 4

It follows that a Hagge circle generated by T can be drawn, using the points D', E', F' and if its centre is denoted by O', we know from the theory of Hagge circles that HTgOO' is a parallelogram. It follows, since Tg = P has co-ordinates (p, 0) that O' has co-ordinates

$$(2a/(1 + a^2) + 2b/(1 + b^2) + 2c/(1 + c^2) - p,\ (1 - a^2)/(1 + a^2) + (1 - b^2)/(1 + b^2) + (1 - c^2)/(1 + c^2)). \quad (4.5)$$

Our proposition that the direct similarity between the circle, centre O through P, and the Hagge circle, centre O', is correct, is confirmed if the 180^o rotation of Q about R, the mid-point of OO', takes Q to T. And indeed the co-ordinates of O' minus the co-ordinates of Q do coincide with those of T, given in Equations (4.1) and (4.2).

5. Further Observations

It is also the case that O' is the centre of circle DEF and that the isogonal conjugate of Q with respect to triangle DEF lies on the Hagge circle centre O'. The circle DEF is therefore directly similar to circle ABC and the circle centre O through P is the Hagge circle of Q with respect to triangle DEF. See Figure 4, where the full symmetry of the construction is finally revealed.

We observe that DD', EE', FF' do concur at a point S on the circumcircle. Its co-ordinates are too complicated to record.

Another rather curious result, as David Monk [3] pointed out, is that the centroid of triangle TPQ coincides with that of ABC. Using P to mean the vector **OP** etc., the proof of this is that, since T + Q = 2R = O', it follows that T + P + Q = O' + P = H = A + B + C.

6. Conjecture supported by *Cabri*

We conclude with a conjecture. As we have seen circles centre O carry triangles that are directly similar to Hagge circles by 180^{o} rotation about the midpoint of the line OO', where O' is the centre of the Hagge circle.

We now conjecture that any circle and the triangles produced on it by a direct congruence of a Hagge circle and its triangles have additional properties that make the matter interesting and significant. We now refer to Figure 5 and the points have meanings as attached to this diagram.
As can be seen the point of rotation of the Hagge circle R is chosen arbitrarily and the angle of rotation is also arbitrary. *Cabri* is so accurate that we have no doubt the conjectures we describe are true. We have not proved them as it seems that algebra is not the medium for doing so, and it seems unlikely that we could put through a proof algebraically anyway.

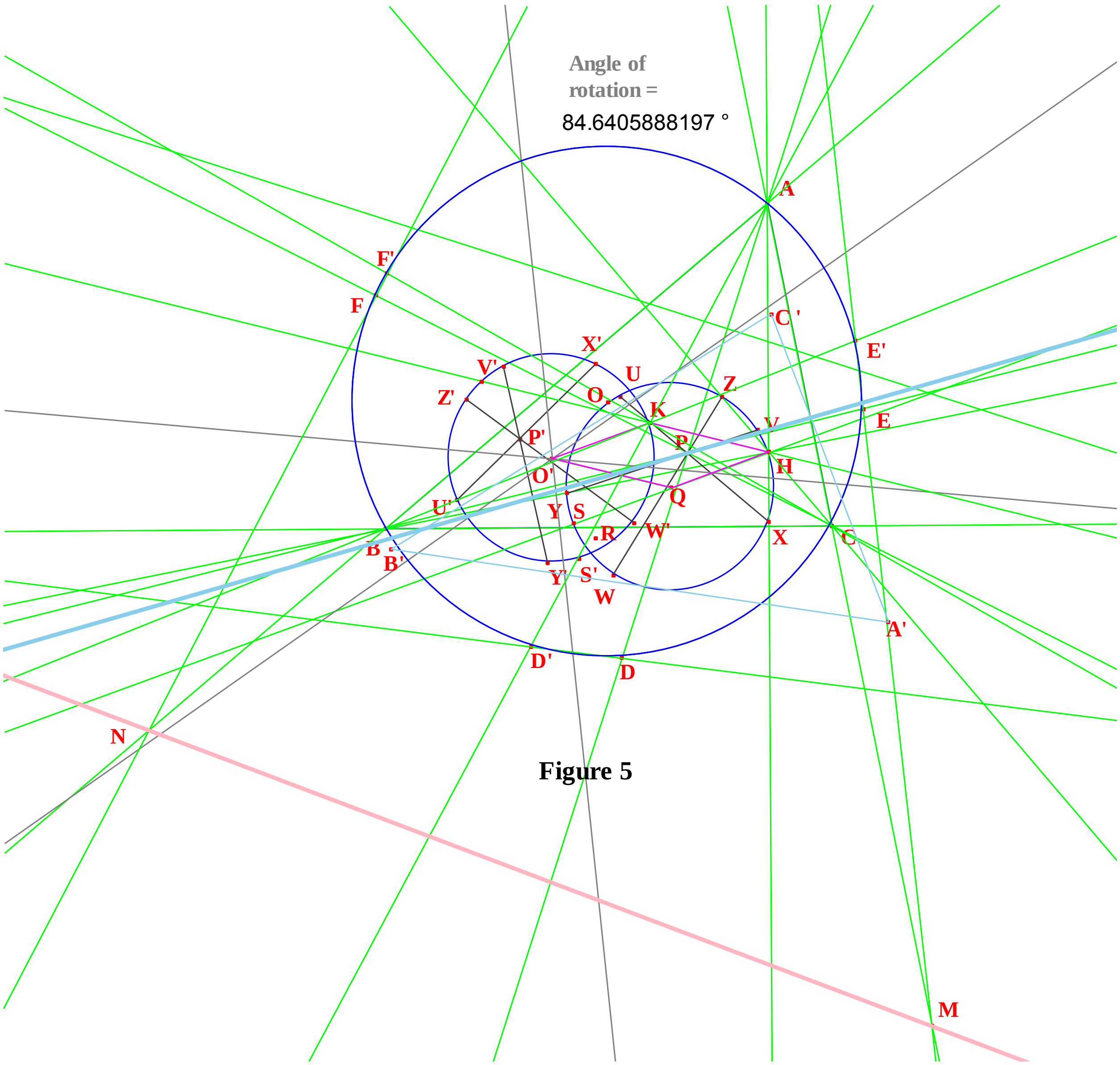

Figure 5

In the Figure the point P generates the Hagge circle in the usual way with U, V, W the reflections of D, E, F in the lines BC, CA, AB respectively, where D, E, F are the intersections of AP, BP, CP with the circumcircle. X, Y, Z are the points on AH. BH, CH lying on circle UVWH. Q is the centre of the Hagge circle and S is the opposite end of the diameter HQ. The direct congruence of the Hagge circle by rotation though R is the circle, centre O', with images U', V', W', X', Y', Z'. Also S' is the image of S. In the case of the circle centre O, the fourth vertex K of the parallelogram O'QHK is the isogonal conjugate of P. The conjecture is that this point K is now a conjugate of P in the sense that if AK, BK, CK meet the circumcircle of ABC at D', E', F' respectively then if L, M, N are the intersections of DD', EE', FF' with BC, CA, AB respectively, L, M, N are collinear. (In the case of the circles centre O the conjugate is the isogonal conjugate and LMN is the line at infinity.) Finally if S' is the image of S under the direct congruence, then X', Y', Z' are now reflections of the point S' in axes (dotted in Figure 5), that make angles 86.405... ^{o} with lines parallel to the altitudes of triangle ABC. (That part of the conjecture is obvious.) In Figure 5 the axis through P of indirect similarity relating ABC and XYZ is shown. The conjugation and the role of the point S at the other end of the diameter to H in the Hagge circle are the matters that require further investigation, but which we do not intend to pursue further.

References

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3. D. Monk, *private communication*.

Flat 4
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP