

A Pair of Isogonal Conjugates produce a Conic through 6 Points

Christopher Bradley

Abstract: If P and Q are an isogonal conjugate pair then circles APQ , BPQ , CPQ intersect the sides of triangle ABC in six points and it is found that a conic passes through them. The construction only produces a conic when P and Q are isogonal conjugates. When P and Q are the Brocard points then the conic degenerates into a pair of lines.

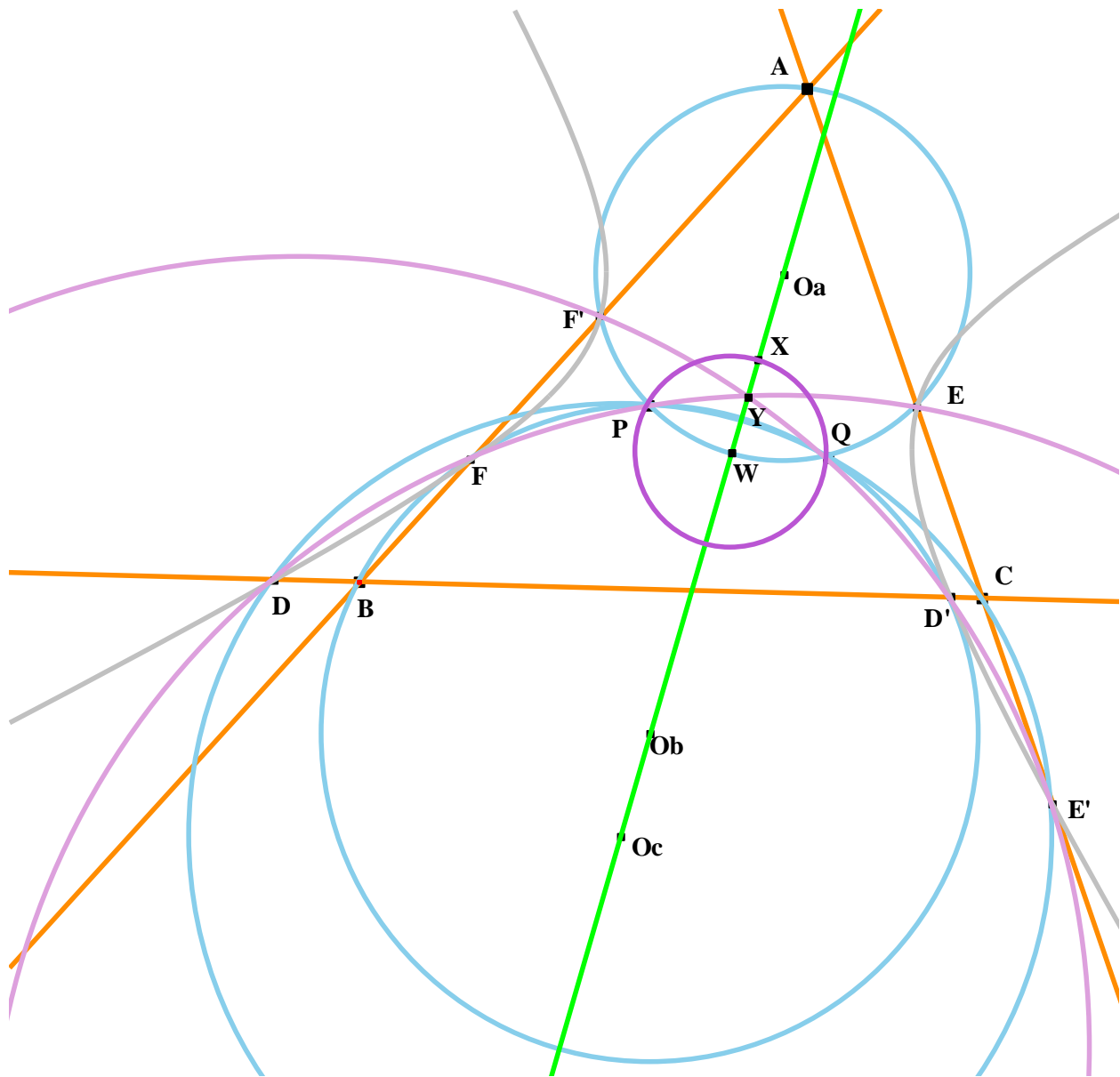


Fig. 1

1. Points P and Q and Circle APQ

We use areal co-ordinates throughout this article. We label P by the co-ordinates (f, g, h) and Q, its isogonal conjugate, by $(a^2/f, b^2/g, c^2/h)$.

In areals a circle has an equation of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(lx + my + nz) = 0, \quad (1.1)$$

where l, m, n are to be found by using the co-ordinates of the three given points lying on the circle.

We find that the circle APQ has

$$l = 0,$$

$$m = (1/k)(c^2g(a^4g^2h^2 + a^2h(2c^2fg^2 - b^2h(f^2 + 2hf + g^2 + 2gh + h^2))) + f^2(b^2h + c^2g)^2), \quad (1.2)$$

$$n = -(1/k)(b^2h(a^4g^2h^2 + a^2g(2b^2fh^2 - c^2g(f^2 + 2fg + g^2 + 2gh + h^2))) + f^2(b^2h + c^2g)^2),$$

and

$$k = (f + g + h)(a^2gh + f(b^2h + c^2g))(b^2h^2 - c^2g^2). \quad (1.3)$$

2. The six points D, D', E, E', F, F' and the conic through these points

Circle APQ meets CA at E with co-ordinates

$$x = h(a^4g^2h^2 + a^2g(2b^2fh^2 - c^2g(f^2 + 2fg + g^2 + 2gh + h^2))) + f^2(b^2h + c^2g)^2, \quad (2.1)$$

$$y = 0,$$

$$z = (a^2gh^2 + f(b^2h^2 - c^2g(f + g)))(g + h)(b^2h + c^2g) - a^2gh$$

Circle APQ meets AB at F' with co-ordinates

$$x = g(a^4g^2h^2 + a^2h(2c^2fg^2 - b^2h(f^2 + 2hf + g^2 + 2gh + h^2))) + f^2(b^2h + c^2g)^2, \quad (2.2)$$

$$y = (a^2g^2h - f(b^2h(f + h) - c^2g^2))(g + h)(b^2h + c^2g) - a^2gh,$$

$$z = 0.$$

The co-ordinates of F, D', D, E' follow from Equations (2.1) and (2.2) by cyclic changes of x, y, z and a, b, c and f, g, h. (F and D from (2.1) and D' and E' from (2.2)).

It may now be shown that these six points lie on a conic. Unfortunately its equation took up to three pages of *DERIVE* output and so is impossible to copy down without error. I regret it must be left to the reader.

Flat 4, Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.