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Between the Triplicate Ratio Circle and the Circumcircle

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Abstract: When UVW is a triangle homothetic (and similar) to triangle ABC through K then the six points where UV, VW, WU intersect the sides of ABC lie on a circle. When UVW becomes a point at K one gets the Triplicate Ratio Circle and when U, V, W reach A, B, C one gets the Circumcircle. The centres O' of these circles move along the Brocard axis.

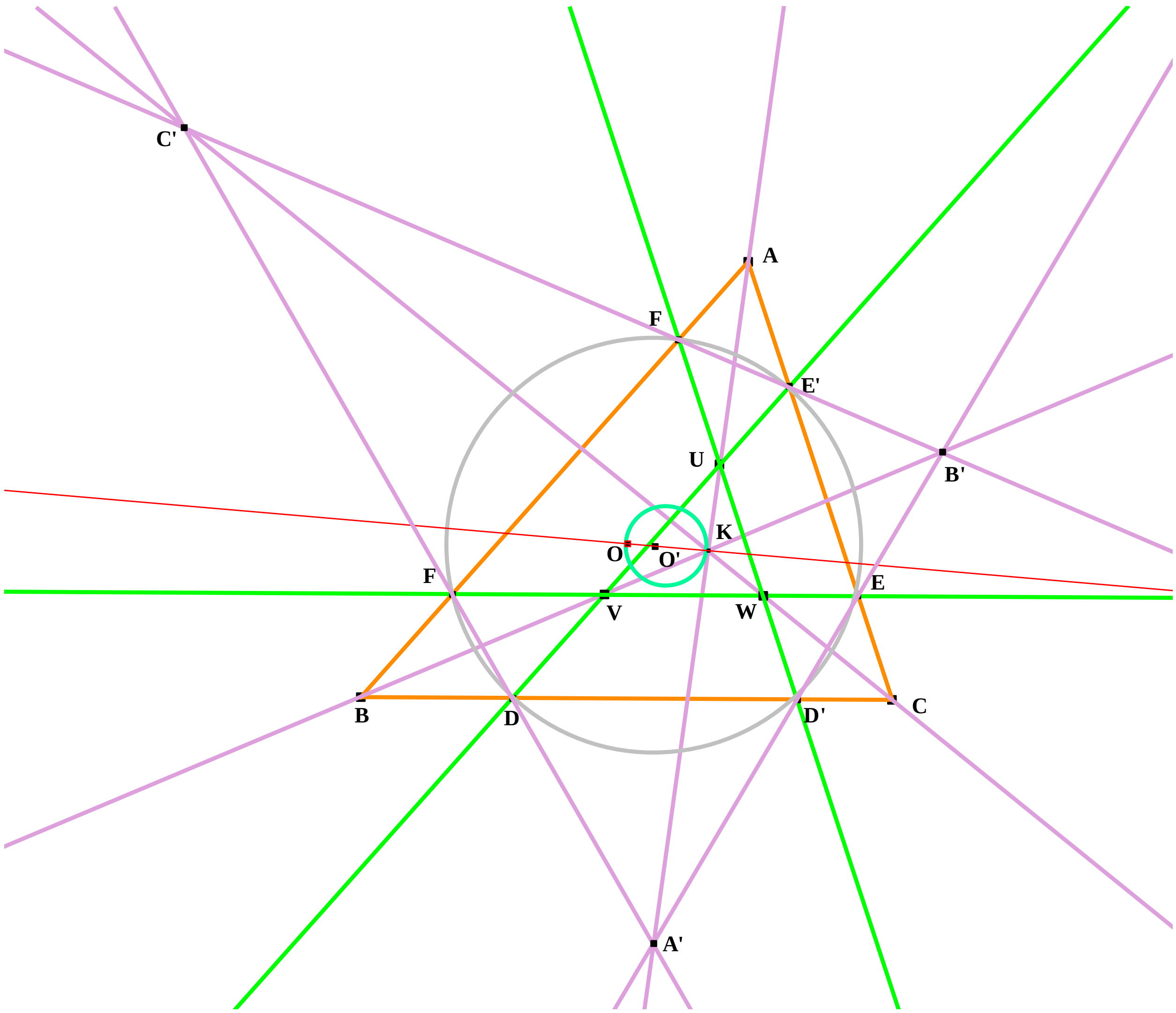

Fig. 1
Generating circles from triangle UVW

1. Introduction

In this article we use areals throughout with ABC as triangle of reference. Point K is the symmedian point with co-ordinates (a^2, b^2, c^2). For simplicity of working we take U, V, W to be the mid- points of AK, BK, CK respectively, but the main result is true whenever AU/UK = BV/VK = CW/WK. This main result is that the six points where UV, VW, WU meet the sides of ABC lie on a circle with centre on the Brocard axis OK, where O is the circumcentre of ABC.

2. The six points D, E, F, D', E', F'

The midpoint U of AK has co-ordinates (x, y, z), where

$$x = (2a^2 + b^2 + c^2)/(2(a^2 + b^2 + c^2)),\ y = b^2/(2(a^2 + b^2 + c^2)),\ z = c^2/(2(a^2 + b^2 + c^2)), \qquad (2.1)$$

The co-ordinates of V, W may be obtained from (2.1) by cyclic change of both x, y, z and a, b, c.

The equation of UV is

$$c^2(x + y) = (2a^2 + 2b^2 + c^2)z. \qquad (2.2)$$

The equations of VW and WU may be obtained from (2.2) again by cyclic change of both x, y, z and a, b, c.

The line UV meets BC at D $(0, 2a^2 + 2b^2 + c^2, c^2)$ and UV meets CA at E'$(2a^2 + 2b^2 + c^2, 0, c^2)$.

Similarly point E has co-ordinates $(a^2, 0, 2b^2 + 2c^2 + a^2)$ and point F' has co-ordinates $(a^2, 2b^2 + 2c^2 + a^2, 0)$. And finally F has co-ordinates $(2c^2 + 2a^2 + b^2, b^2, 0)$ and D' has co-ordinates $(0, b^2, 2c^2 + 2a^2 + b^2)$.

3. The circle DEFD'E'F'

It may now be shown that the six points D, E, F, D', E', F' lie on the circle

$$a^2yz + b^2zx + c^2xy + (x + y + z)(lx + my + nz) = 0, \qquad (3.1)$$

where

$$l = (1/(4(a^2 + b^2 + c^2)^2)(b^2c^2(a^2 + 2b^2 + 2c^2))\ , \qquad (3.2)$$

and m, n follow from (3.2) by cyclic change of a, b, c.

The centre O' of this circle has x-co-ordinate the complicated expression

$$x = -2a^2(2a^{10} + 5a^8(b^2 + c^2) + 2a^6(b^4 + b^2c^2 + c^4) - 2a^4(2b^6 + 9b^4c^2 + 9b^2c^4 + 2c^6)$$

$$– 2a^2(2b^8 + 11b^6c^2 + 18b^4c^4 + 11b^2c^6 + 2c^8) – b^{10} – c^2(7b^8 + 16b^6c^2 + 16b^4c^4 + 7b^2c^6 + c^8)). \quad (3.3)$$

The y- and z- co-ordinates follow from (3.3) by cyclic change of a, b, c.

It may now be shown that O' lies on the Brocard axis OK with equation

$$b^2c^2(b^2 – c^2)x + c^2a^2(c^2 – a^2)y + a^2b^2(a^2 – b^2)z = 0. \quad (3.4)$$

As U, V, W move along AK, BK, CK in harmony so O' moves along OK, being at O when U, V, W tend to A, B, C and at the mid point of OK when U, V, W all tend to K.

4. The points A', B', C'

A further property is worth mentioning and that is that DF', ED' and AK concur at a point A' and similarly for points B' and C'.

Point A' has co-ordinates (x, y, z), where

$$x = – a^2(2a^2 + b^2 + c^2),\ y = b^2(a^2 + 2b^2 + 2c^2),\ z = c^2(a^2 + 2b^2 + 2c^2), \quad (4.1)$$

and where B', C' have co-ordinates which may be written down from (4.1) by cyclic change of x, y, z and a, b, c. It follows that triangles ABC and A'B'C' are in perspective with vertex K.

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