Article 2

The Four Hagge circles

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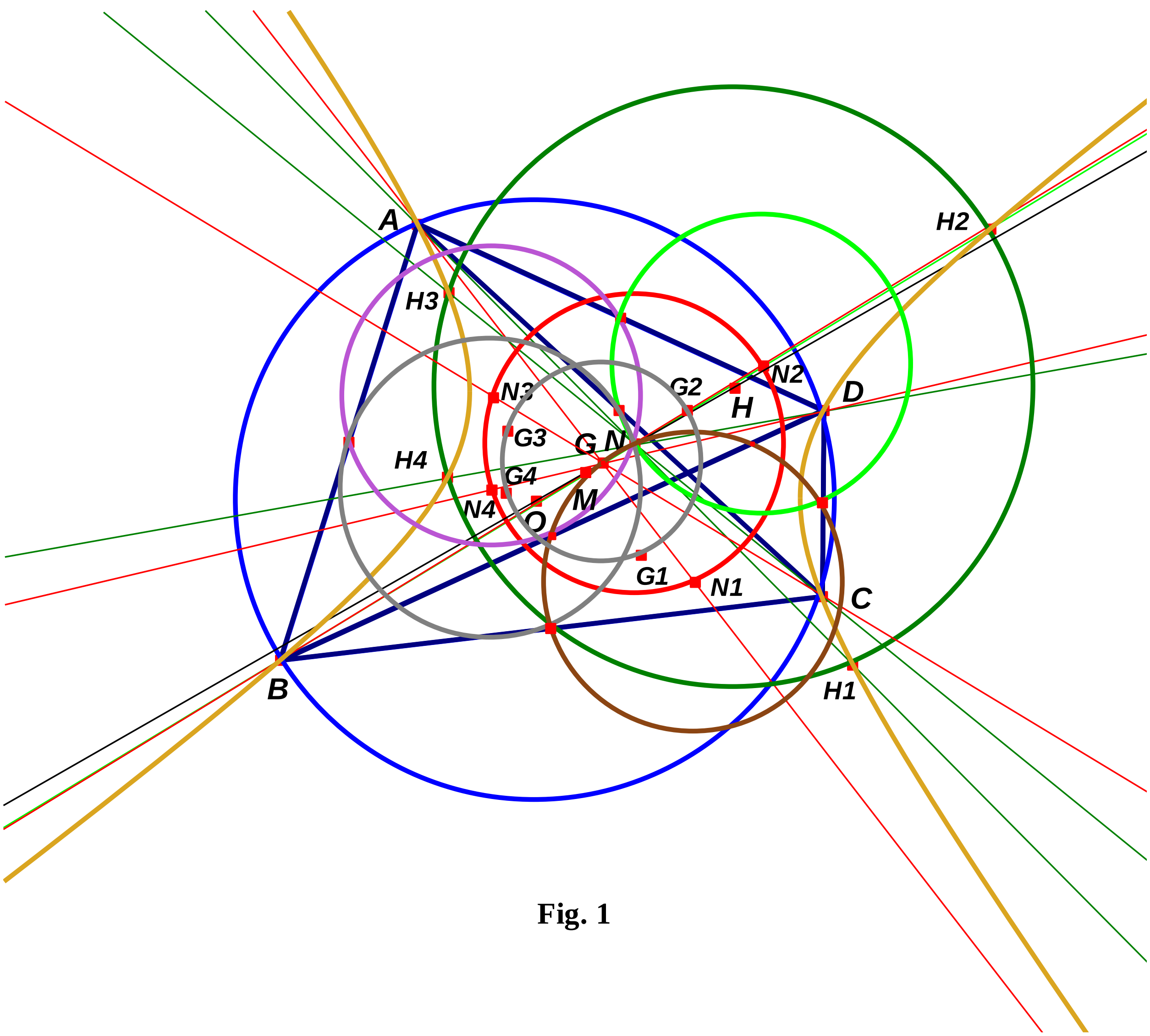

Fig. 1

1. Preliminary analysis

In later sections we state and prove a number of theorems about the four Hagge circles, with respect to an appropriately chosen centre of inverse similarity *P*, of the four triangles *BCD*, *ACD*, *ABD*, *ABC* that comprise a cyclic quadrilateral *ABCD*. It is desirable therefore to give a description of the basic configuration involved and to prove some straightforward facts that are required later.

We use vectors with origin *O*, the centre of the circumscribing circle, and with *A*, *B*, *C*, *D* having vector positions **a**, **b**, **c**, **d** each of which has magnitude *R*, the radius of the circle. The triangles *BCD*, *ACD*, *ABD*, *ABC* are denoted by Δ_1, Δ_2, Δ_3, Δ_4 respectively. Suffices are used for key points in these triangles. Thus H_k is the orthocentre of Δ_k, N_k is the nine-point centre of Δ_k and G_k is the centroid of Δ_k, $k = 1, 2, 3, 4$. The vector positions corresponding to the points are $\mathbf{h_1} = \mathbf{b} + \mathbf{c} + \mathbf{d}$, $\mathbf{h_2} = \mathbf{a} + \mathbf{c} + \mathbf{d}$, $\mathbf{h_3} = \mathbf{a} + \mathbf{b} + \mathbf{d}$, $\mathbf{h_4} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ and, as in any triangle, $\mathbf{n_k} = ½\mathbf{h_k}$ and $\mathbf{g_k} = ⅓\mathbf{h_k}$. The following facts are now easy to establish:

(i) AH_1, BH_2, CH_3, DH_4 are concurrent at the point *N* with vector position $½(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$;

(ii) AN_1, BN_2, CN_3, DN_4 are concurrent at the point *G* with vector position $⅓(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$;

(iii) AG_1, BG_2, CG_3, DG_4 are concurrent at a point *M* with vector position $¼(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$;

(iv) H_1, H_2, H_3, H_4 lie on a circle of radius *R* with centre *H*, which has position vector $(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$ and moreover $H_1H_2H_3H_4$ is the image of *ABCD* under a180° rotation about *N*;

(v) N_1, N_2, N_3, N_4 lie on a circle of radius $½R$ with centre *N* and $N_1N_2N_3N_4$ is homothetic with $H_1H_2H_3H_4$, with centre *O* and enlargement factor ½;

(vi) The nine-point circles of triangles Δ_1, Δ_2, Δ_3, Δ_4 , each having radius $½R$ therefore all pass through *N*;

(vii) A known result is that a rectangular hyperbola through the vertices of a triangle passes through it orthocentre. It follows that the rectangular hyperbola Σ through *A*, *B*, *C*, *D* passes through all of H_1, H_2, H_3, H_4;

(viii) Another known result is that the centre of a rectangular hyperbola passing through the vertices of a triangle lies on its nine-point circle. In consequence of (vii) the centre of Σ is the point *N*;

(ix) G_1, G_2, G_3, G_4 lie on a circle of radius $⅓R$ with centre *G*, which has position vector $⅓(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$;

(x) The points *O*, *M*, *G*, *N*, *H* are collinear and if *O* and *H* are given co-ordinates 0, 1 on this line, then *M*, *G*, *N* have co-ordinates ¼, ⅓, ½ respectively.

See Fig. 1 for an illustration of these results.

2. Setting the scene

It is clear from Speckman's work on indirect similar perspective triangles that if we now choose any point P on the rectangular hyperbola defined in Section 1 Result (vii), it is possible to find the Hagge circle of P with respect to each of the four triangles Δ_1, Δ_2, Δ_3, Δ_4 and to draw them on the same diagram. These are the four Hagge circles Γ_1, Γ_2, Γ_3, Γ_4 of the article heading and they are shown in Fig. 2. Note that as the axes of inverse similarity through P are parallel to the asymptotes of the rectangular hyperbola Σ, these axes coincide for each of the Hagge circles.

A consistent notation is essential and this we now describe. See Fig. 2. The centres of the circles are denoted by Q_k, k = 1, 2, 3, 4. For reasons that become clear shortly we re-label the four orthocentres A_1, B_2, C_3, D_4, so that, for example D_4 is the orthocentre of triangle ABC. The lines AP, BP, CP, DP meet the circumcircle Γ at points A', B', C', D'. We now consider the labelling of points on the Hagge circle Γ_4, which is the Hagge circle of P with respect to triangle ABC. They all carry the subscript 4. The reflection of A' in BC we denote by A_4', the reflection of B' in CA we denote by B_4' and the reflection of C' in AB we denote by C_4'. These replace the labels U, V, W used in Article 1. The points where $A_4'P$, $B_4'P$, $C_4'P$, D_4P meet Γ_4 are denoted by A_4, B_4, C_4, D_4' respectively. The first three of these replace the labels X, Y, Z in Article 1. Points with suffices 1, 2, 3 are similarly defined on circles Γ_1, Γ_2, Γ_3. The labels Pg_1, Pg_2, Pg_3, Pg_4 are given to the isogonal conjugates of P with respect to triangles Δ_1, Δ_2, Δ_3, Δ_4 respectively. The double lines of inverse symmetry through P are denoted by L and L'. The centre of the rectangular hyperbola Σ is denoted by M.

In Section 3 we use Cartesian co-ordinates, origin M and the asymptote parallel to L is chosen as the x-axis. In this way Σ has equation $xy = 1$ (by choice of scale), and we use a parameter t on the hyperbola so that its points have co-ordinates $(t, 1/t)$. The points A, B, C, D, P are given parameters a, b, c, d, p, where it is known, since $ABCD$ is cyclic, that $abcd = 1$ and the parameters of A_1, B_2, C_3, D_4, since they are the orthocentres, have parameters $-a$, $-b$, $-c$, $-d$ respectively.

The following theorems now hold:

Theorem 1
The centres, Q_1, Q_2, Q_3, Q_4, of the four Hagge circles are collinear.

Theorem 2
The points A_1, A_2, A_3, A_4, A_1', A_2', A_3', A_4' are collinear and similarly for the points B_k, B_k', k = 1, 2, 3, 4 and C_k, C_k', k = 1, 2, 3, 4 and D_k, D_k', k = 1, 2, 3, 4.

Theorem 3

The quadrilateral $Pg_1Pg_2Pg_3Pg_4$ is similar to the quadrilateral $A'B'C'D'$.

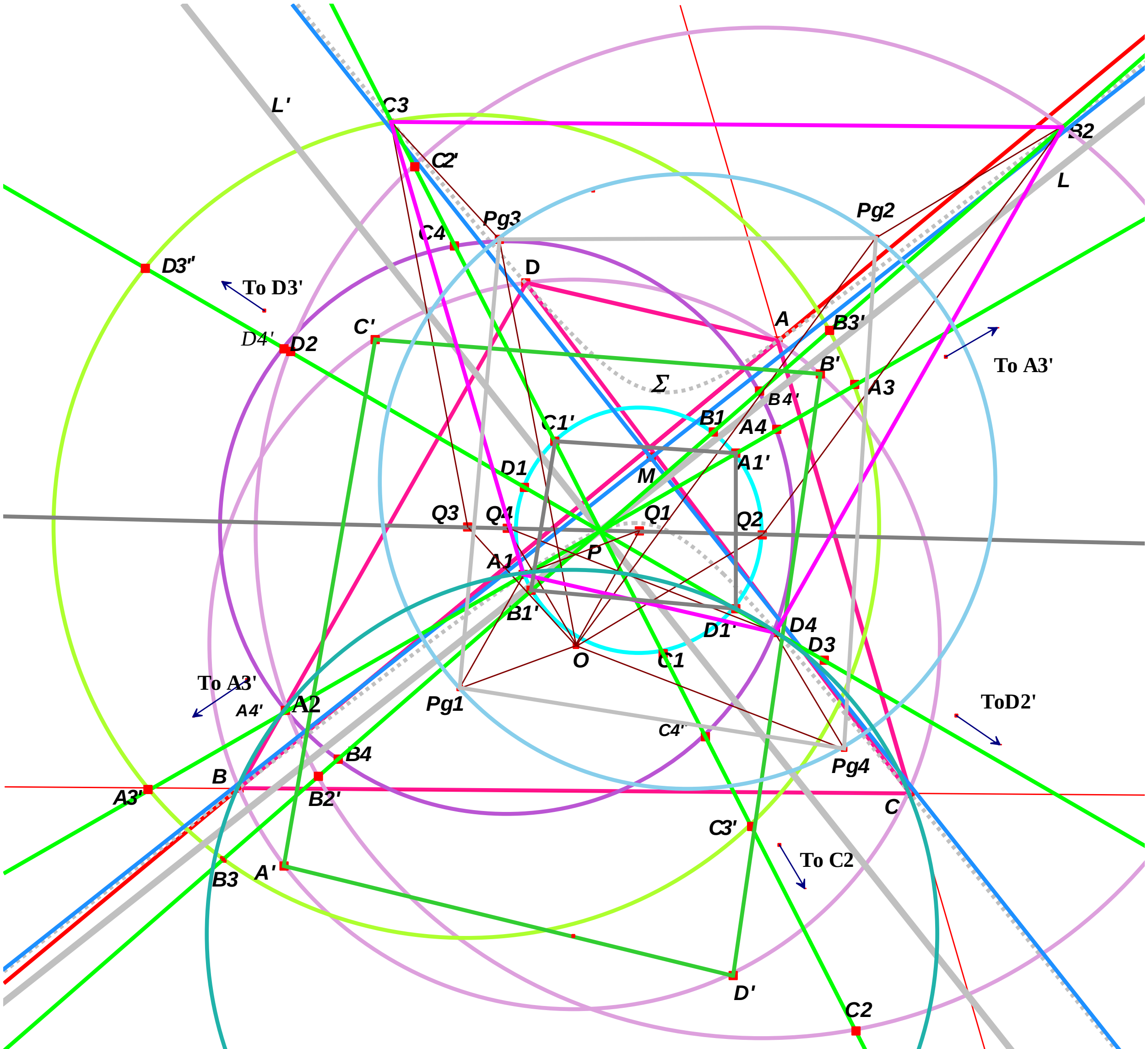

Fig. 2

3. The analysis

The co-ordinates of *O*

The chord AB has equation

$$aby + x = a + b. \quad (3.1)$$

The midpoint of AB has co-ordinates $(\frac{1}{2}(a + b), \frac{1}{2}(a + b)/(ab))$. It follows that the perpendicular bisector of AB has equation

$$2ab(y - abx) = (a + b)(1 - a^2b^2). \qquad (3.2)$$

The perpendicular bisector of BC has equation

$$2bc(y - bcx) = (b + c)(1 - b^2c^2).$$

These meet at O, the centre of Γ, at the point with co-ordinates $(\frac{1}{2}(a + b + c + d), \frac{1}{2}(1/a + 1/b + 1/c + 1/d)$. Here we have used $abcd = 1$.

It can be shown that the radius of the circumcircle is

$$\tfrac{1}{2}\sqrt{(a^2 + b^2 + c^2 + d^2 + 1/a^2 + 1/b^2 + 1/c^2 + 1/d^2)}. \qquad (3.3)$$

The equation of the circumcircle Γ

From the above information this is easily proved to be

$$2x^2 - 2x(a + b + c + d) + 2y^2 - 2y(1/a + 1/b + 1/c + 1/d) + (ab + ac + ad + bc + bd + cd + 1/ab + 1/ac + 1/ad + 1/bc + 1/bd + 1/cd) = 0. \qquad (3.4)$$

Alternatively, in terms of a, b, c only it has the form

$$abcx^2 - (abc(a + b + c) + 1)x + abcy^2 - (a^2b^2c^2 + ab + bc + ca)y + (abc)(ab + bc + ca) + a + b + c = 0. \qquad (3.5)$$

A digression

The reflection of the point with co-ordinates (f, g) in the line with equation $lx + my = n$ is the point with co-ordinates (h, k) where

$$h = \{f(m^2 - l^2) + 2l(n - gm)\}/(l^2 + m^2),$$
$$k = \{2mn + g(l^2 - m^2) - 2flm\}/(l^2 + m^2).$$

This is straightforward and is left to the reader. This is used later to reflect $ABCD$ in the line L and then dilate through P to obtain the Hagge circle $A_4B_4C_4D_4$.

The indirect similarity

We now determine the equations governing the indirect similarity which maps the circumcircle into the Hagge circle of P with respect to Δ_4. This is done by following what happens to the point $D(d, 1/d)$. We know the image of this point is the orthocentre $D_4(-1/abc, -abc)$ and we know the mapping is effected by a reflection in the line L, with equation $y = 1/p$ followed by an enlargement (reduction) by a factor PD_4/PD. As far as the reflection is concerned we may use the analysis of the last paragraph with $l = 0$, $m = 1$, $n = 1/p$. The result of this reflection on D is to produce the point with co-ordinates $(d, 2/p – 1/d)$.

The enlargement (reduction) factor (comparing x-co-ordinates of the points concerned) is equal to $(p + d)/(p – d) = (abcp + 1)/(abcp – 1)$. It may now be checked that the reflection in L followed by the dilation with this enlargement (reduction) factor takes the point with co-ordinates (h, k) to the point with co-ordinates (x, y) where

$$x = (abchp – 2p + h)/(abcp – 1), \quad (3.6)$$

and

$$y = (abc(kp – 2) + k)/(1 – abcp). \quad (3.7)$$

Using $h = a$, $k = 1/a$ we deduce the co-ordinates of A_4 to be

$$((a^2bcp + a – 2p)/(abcp – 1), (2a^2bc – abcp – 1)/(a(abcp – 1))),$$

with similar expressions for the co-ordinates of B_4 and C_4 by using b and c, in x and y above, instead of a.

The equations of the Hagge circles

From here it is quite an elaborate calculation to obtain the equation of the Hagge circle $A_4B_4C_4$ and check the fundamental theorem that D_4 lies on this circle. The computer algebra package *DERIVE* was used to perform the calculations, and the result is that Γ_4 has equation

$$\begin{aligned} &abc(abcp – 1)(x^2 + y^2) – (a^2b^2c^2p(a + b + c) + abc(a + b + c) – 3abcp + 1)x \\ &\quad + (a^3b^3c^3p – 3a^2b^2c^2 + (abcp + 1)(bc + ca + ab))y \\ &\quad – 2a^3b^3c^3 + a^2b^2c^2p(a + b + c) + abc(bc + ca + ab) – (abcp + 1)(a + b + c) – 2p = 0. \quad (3.8) \end{aligned}$$

The equations of the other Hagge circles follow immediately by using other triplets of parameters instead of a, b, c. The co-ordinates of the centres of these circles may now be written down and it has been checked, using *DERIVE*, that any three of the four centres are collinear. Theorem 1 is now proved, but it also follows from a geometrical argument. The centre O of the circumcircle is the common circumcentre of all four triangles Δ_1, Δ_2, Δ_3, Δ_4. If we now reflect O in the two lines of inverse similarity, it maps into a pair of points collinear with P. Since the centres of the four Hagge circles are now the images of these points by means of dilations through P with enlargements (reductions) using different factors, the resulting images all lie on this line, and this line also passes through P. Theorem 2 follows by a similar argument, bearing in mind that we already know primed and unprimed pairs of points such as A_4, A_4' are collinear with P.

The quadrilateral *A'B'C'D'*

The equation of the line AP is

$$x + apy = a + p. \quad (3.9)$$

This meets the circumcircle again at the point A' with co-ordinates (x, y) where

$$x = \{ap^2(abc(b + c) + 1) – p(a^2b^2c^2 + a(b + c) – bc) + abc\}/\{bc(a^2p^2 + 1)\}, \quad (3.10)$$

$$y = \{abcp^2 + p(a^2bc – abc(b + c) – 1) + ab^2c^2 + b + c\}/\{bc(a^2p^2 + 1)\}. \quad (3.11)$$

The point D' has co-ordinates similar to these, but with d replacing a.
From these co-ordinates we can work out $(A'D')^2$ and the result is

$$\{(a – d)^2(b – p)^2(c – p)^2(b^2c^2 + 1)\}/\{b^2c^2(a^2p^2 + 1)(d^2p^2 + 1)\}. \quad (3.12)$$

The quadrilateral $Pg_1Pg_2Pg_3Pg_4$

The co-ordinates (h, k) of Pg_4, the isogonal conjugate of P with respect to triangle ABC are given by

$$h = (a + b + c – p)/(1 – abcp), \quad (3.13)$$

$$k = (p(bc + ca + ab) – abc)/(abcp – 1). \quad (3.14)$$

This may be checked as follows: Let the line from A to (h, k) meet Γ at A'', then it is easy to show that $A'A''$ is parallel to BC. The symmetry of h, k with respect to a, b, c now proves that (h, k) are the co-ordinates of Pg_4, as this calculation reflects one of the standard constructions for an isogonal conjugate. Indeed the parallel $A'A''$ to BC indicates that AP and APg_4 are reflections of each other in the internal bisector of angle A. The co-ordinates of Pg_1 now follow from (h, k) by exchanging a and d.

From these co-ordinates we can work out $(Pg_1Pg_4)^2$ and the result is

$$\{(a – d)^2(b^2c^2 + 1)(b^2p^2 + 1)(c^2p^2 + 1)\}/\{(abcp – 1)^2(bcdp – 1)^2\}. \quad (3.15)$$

The similarity of the quadrangles

The ratio of the squares of the side lengths and diagonals of the quadrangles $Pg_1Pg_2Pg_3Pg_4$ and $A'B'C'D'$ may now be calculated and is the totally symmetric expression

$$\{(a^2p^2 + 1)(b^2p^2 + 1)(c^2p^2 + 1)(d^2p^2 + 1)\}/\{(a – p)^2(b – p)^2(c – p)^2(d – p)^2\}. \quad (3.16)$$

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