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The Two Central Lines in a Cyclic Quadrilateral

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Abstract: There are five main central points in a cyclic quadrilateral: the circumcentre O, the centroid F, the centre of mass G, the intersection of the diagonals E and the intersection of the lines joining the midpoints of opposite sides T. It is proved in this article that GTE is a straight line and that OTF is a straight line. Furthermore ET = 3TG and OT = 3TF and so GF is parallel to OE. (The anticentre also lies on OTF.)

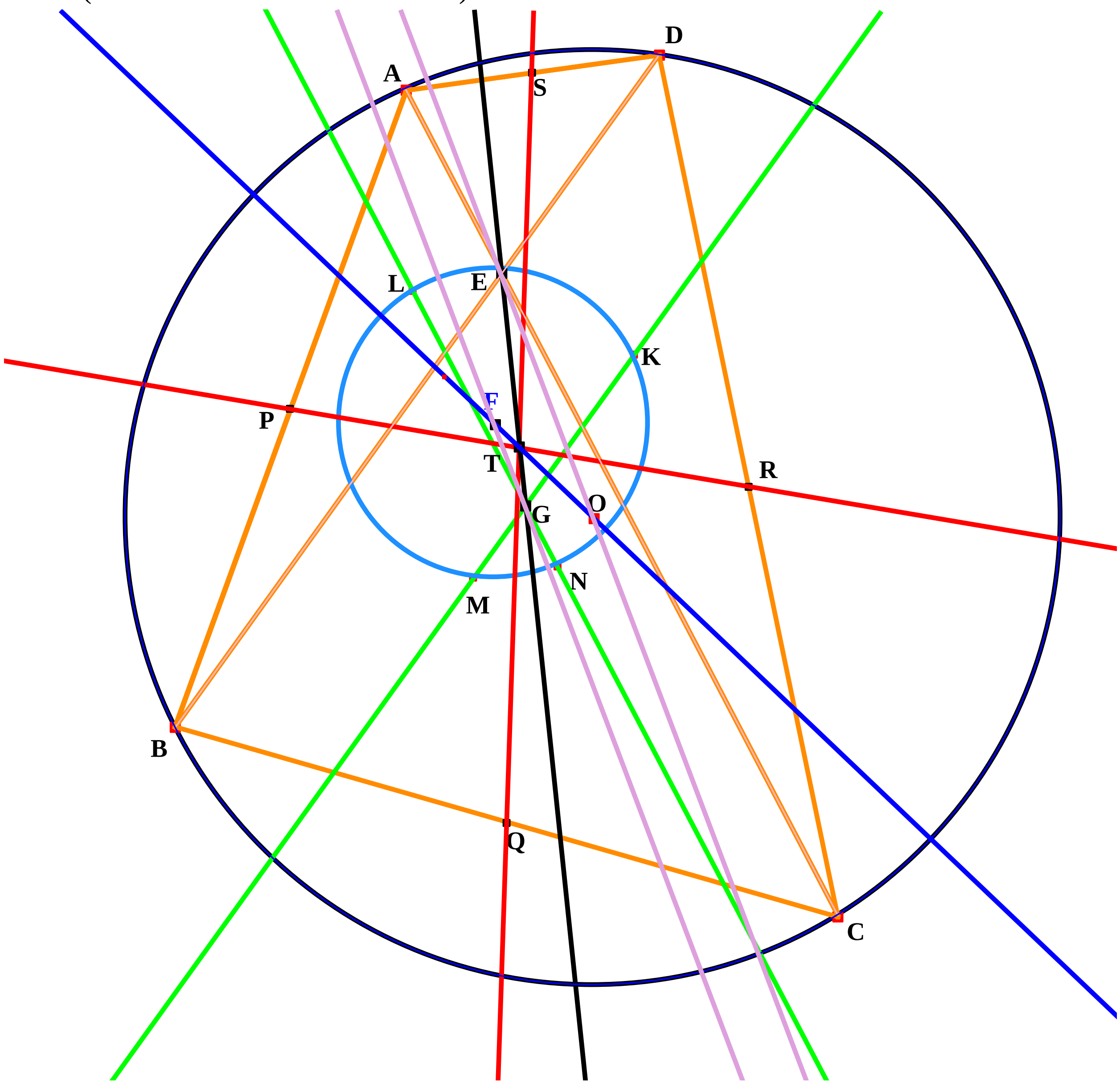

Fig.1
The central lines of a cyclic quadrilateral

1. The centroid F

We use areal co-ordinates with ABC the triangle of reference and D the general point on the circumcircle with co-ordinates (x, y, z), where

$$x = -a^2t(1-t),\ y = b^2(1-t),\ z = c^2t. \tag{1.1}$$

We label the midpoints of AB, BC, CD, DA to be P, Q, R, S respectively. The lines PR QS meet at a point T with normalized co-ordinates (x, y, z), where

$$x = (2a^2t(t-1) + b^2(1-t) + c^2t)/\{4(a^2t(t-1) + b^2(1-t) + c^2t)\}, \tag{1.2}$$
$$y = (a^2t(t-1) + 2b^2(1-t) + c^2t)/\{4(a^2t(t-1) + b^2(1-t) + c^2t)\}, \tag{1.3}$$
$$z = (a^2t(t-1) + b^2(1-t) + 2c^2t)/\{4(a^2t(t-1) + b^2(1-t) + c^2t)\}. \tag{1.4}$$

We label the centroids of triangles DAB. ABC, BCD, CDA to be L, M, N, P respectively. Their normalized co-ordinates are:

L:
$$x = (2a^2t(t-1) + b^2(1-t) + c^2t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\},$$
$$y = (a^2t(t-1) + 2b^2(1-t) + c^2t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\}, \tag{1.5}$$
$$z = c^2t/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\}.$$

M:
$$x = y = z = 1/3. \tag{1.6}$$

N:
$$x = a^2t(t-1)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\},$$
$$y = (a^2t(t-1) + 2b^2(1-t) + c^2t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\}, \tag{1.7}$$
$$z = (a^2t(t-1) + b^2(1-t) + 2c^2t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\}.$$

P:
$$x = (2a^2t(t-1) + b^2(1-t) + c^2t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\},$$
$$y = b^2(1-t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\}, \tag{1.8}$$
$$z = (a^2t(t-1) + b^2(1-t) + 2c^2t)/\{3(a^2t(t-1) + b^2(1-t) + c^2t)\}.$$

The equation of the circle LMNP is of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0, \tag{1.9}$$

where we find $k = 9(a^2t(t-1) + b^2(1-t) + c^2t)$, and

$$u = (1/k)(a^4t(t-1) - a^2(b^2(t-1)(t+2) + c^2t(t-3)) + 2(b^4(t-1) - 2b^2c^2 - c^4t), \tag{1.10}$$
$$v = -(1/k)(2a^4t(t-1) + a^2(4c^2t^2 - b^2(t-1)(2t+1)) + b^4(t-1) + b^2c^2(1-3t) + 2c^4t), \tag{1.11}$$
$$w = -(1/k)(2a^4t(t-1) + a^2(4b^2(t-1)^2 + c^2t(3-2t)) + (c^2 - b^2)(2b^2(t-1) - c^2t)). \tag{1.12}$$

The centre of this circle is the centroid F which has co-ordinates

$$x = -(a^4 - a^2(b^2 + c^2 + 2u - v - w) + (b^2 - c^2)(v - w)),$$
$$y = a^2(b^2 + w - u) - b^4 + b^2(c^2 - u + 2v - w) + c^2(u - w), \tag{1.13}$$
$$z = a^2(c^2 - u + v) + b^2(c^2 + u - v) - c^4 - c^2(u + v - 2w).$$

Here u, v, w are given by Equations (1.10) – (1.12)

With circumcentre O it is now straightforward to show that O, T and F are collinear and OT = 3TF.

A final note is worth making. If you take the midpoints of the sides P, Q, R, S and drop perpendiculars on to opposite sides they concur at a point that is known by the horrible name 'the anticentre Ă' and it satisfies OT = TĂ. If **a, b, c, d** are the vector positions of A, B, C, D with respect to O then **OT = (**1/4)**(a + b + c + d)** and **OF =** (1/3)(**a + b + c + d)** and **OĂ = (**1/2)(**a + b + c + d).**

It is true that the whole of this Section would have been more easily established using vectors, but this would not be the case for the second central line GTE.

2. The centre of mass G and the line GTE

L, M, N and P are the centres of mass of triangles DAB, ABC, BCD and CDA respectively. It follows that the centre of mass G of ABCD lies on the line LN and also on the line MP and hence G is the point of intersection of LN and MP.

The equation of LN is

$$(a^2t(t-1) + 2b^2(1-t) + c^2t)x - (2a^2t(t-1) + b^2(1-t) + 2c^2t)y + (a^2t(t-1) + 2b^2(1-t) + c^2t)z = 0 \qquad (2.1)$$

The equation of MP is

$$(a^2(t-1) + 2c^2)x + (a^2(t-1) - c^2)y - (2a^2(t-1) + c^2)z = 0. \qquad (2.2)$$

Lines LN and MP meet at the centre of mass G with co-ordinates

$$\begin{aligned} x &= a^4t(t-1)^2 + 2a^2c^2t(t-1) - c^2b^2(t-1) + c^4t, \\ y &= (a^2t(t-1) + 2b^2(1-t) + c^2t)(a^2(t-1) + c^2), \\ z &= a^4t(t-1)^2 + a^2(1-t)(b^2(t-1) - 2c^2t) + c^4t. \end{aligned} \qquad (2.3)$$

Line AC has equation y = 0 and line BD has equation

$$c^2x + a^2(1-t)z = 0. \qquad (2.4)$$

These two lines meet at E with co-ordinates $E(a^2(t-1), 0, c^2)$.

It may now be shown that GTE is a straight line and that ET = 3TG. Unit masses at A, B, C, D do, of course have centre of mass at T. The distinction between these various points is sometimes quite unclear in the literature.

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