Article 19
Circular Perspective

Christopher J Bradley

1. Introduction

If two triangles ABC and PQR are such that AP, BQ, CR are concurrent at a point V then the triangles are said to be in perspective. The vertex V is sometimes called the *perspector*. This relation is obviously a symmetric property. When the triangles are in perspective then the dual property also holds that L = BC^QR, M = CA^RP, and N = AB^PQ are collinear. The line LMN is called Desargues' axis of perspective and more recently is sometimes called the *perspectrix*. The main theorem is that if triangles are in double perspective so that ABC is in perspective, say, with both triangles PQR and QRP, then they are bound to be in triple perspective, meaning that triangle ABC is in perspective with triangle RPQ as well. There is also the concept of *reverse perspective* when one considers ABC in relation to triangles PRQ, RQP or QPR. For example, pairs of triangles in the Brocard porism are in perspective with perspector the symmedian point K and are in triple reverse perspective with the three perspectors lying on the polar line of K.

In this paper we introduce a concept which we call *circular perspective*. This is a relation between a pair of triangles ABC and PQR in which the vertices of both triangles possess the property that vertices of one do not lie on the sides or extended sides of the other. The definition of *circular perspective* is as follows: ABC and PQR are in circular perspective if circles BCP, CAQ and ABR have a common point D. It is by no means obvious that this is a symmetric relation, but we prove that this is the case. In fact we prove two main theorems:

(i) If ABC is in circular perspective with PQR, then PQR is in circular perspective with ABC.

(ii) If ABC is in circular perspective with both PQR and QRP then it is in circular perspective with triangle RPQ.

We conclude our study by investigating an interesting case in the geometry of the triangle, involving the orthocentroidal circle and the Brocard points, in which triple circular perspective exists. Note that there is no concept of reverse perspective since, for example, circle CPQ is the same as circle CQP. The theorem stated in (i) is illustrated in Figure 1.

2. Circular perspective is a symmetrical relation

The proof of (i) is straightforward. If D is the point of concurrence of circles BCP, CAQ and ABR, then invert with respect to D and the configuration becomes one in which (using stars for inverted points) P* lies on B*C*, Q* lies on C*A* and R* lies on A*B*. It then follows that circles Q*R*A*, R*P*B*, P*Q*C* are concurrent at the point S*, the Miquel point. The inverse image S of the point S* is now the point of concurrence of circles QRA, RPB and PQC.

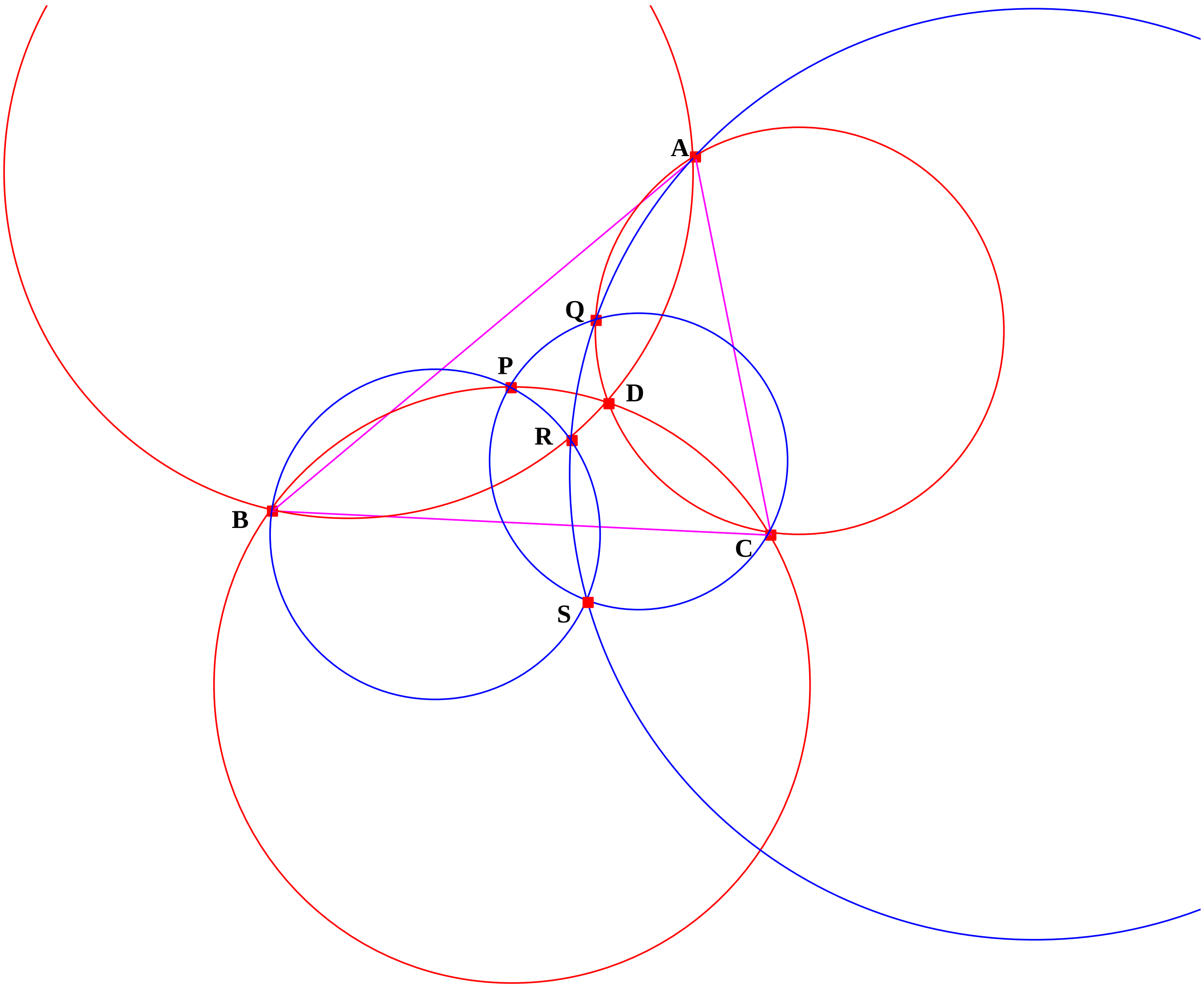

Figure 1

3. Double circular perspective implies triple circular perspective

Suppose now that the circles BCP, CAQ, ABR meet at D and circles BCQ, CAR, ABP meet at E. We want to prove that circles BCR, CAP, ABQ meet at a point F. As in Section 2 we may invert with respect to D, and dropping the stars on invoice points it is sufficient to prove that if P, Q, R lie on BC, CA, AB respectively and circles BCQ, CAR, ABP meet at a point E, then circles BCR, CAP, ABQ meet at a point F. This result is illustrated in Figure 2. We prove this result using areal co-ordinates with ABC as triangle of reference and supposing P has co-ordinates (0, p, 1 – p), Q has co-ordinates (1 – q, 0, q) and R has co-ordinates (r, 1 – r, 0).

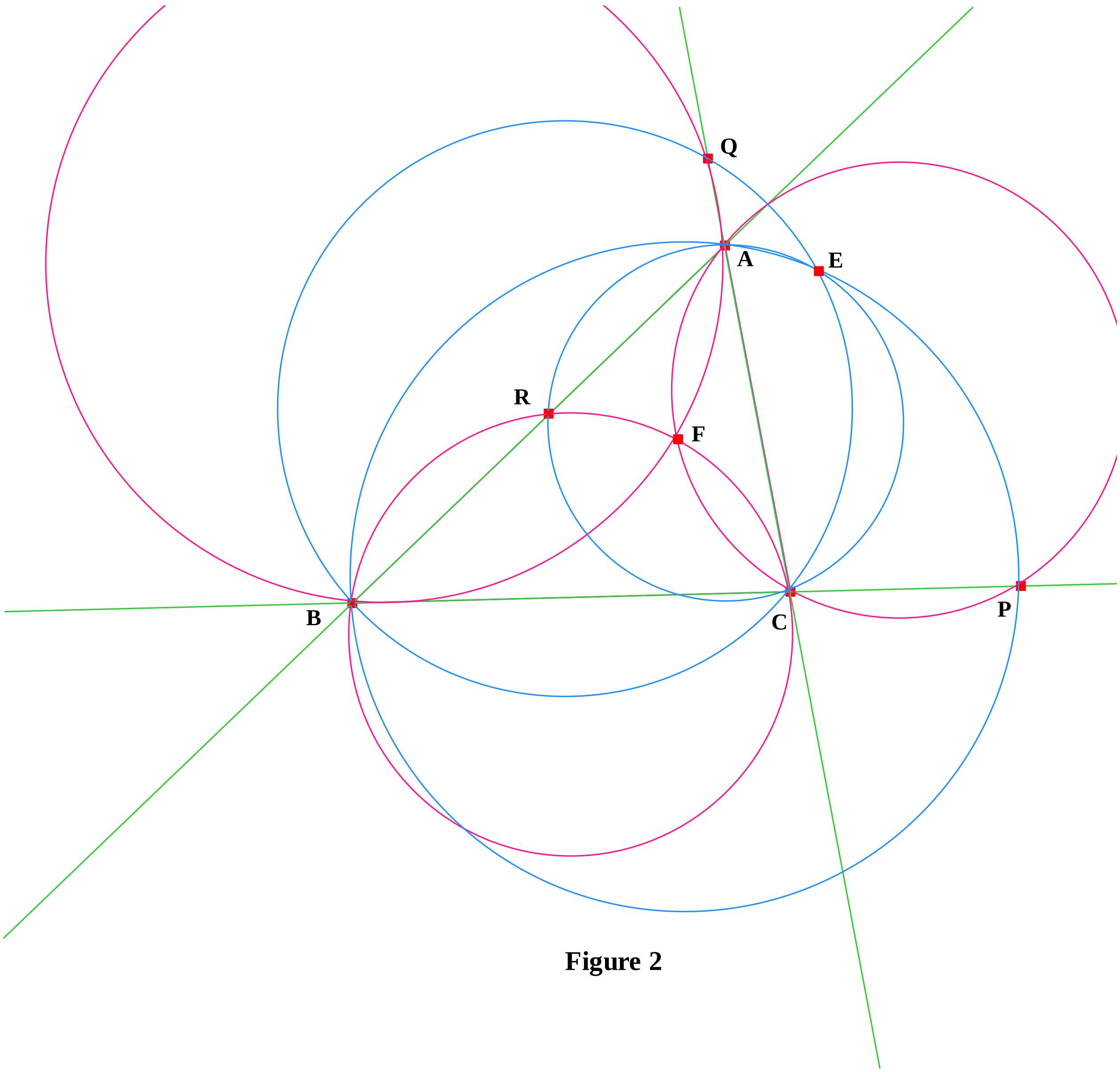

Figure 2

The equation of any circle using areal co-ordinates is

$$a^2yz + b^2zx + c^2xy - (x + y + z)(ux + vy + wz) = 0. \qquad (3.1)$$

To obtain the equation of circle ABP we insert into Equation (3.1) the co-ordinates of A, B, P in succession and obtain three equations for u, v, w. The solution is then substituted back into Equation (3.1). The result is u = v = 0 and w = a^2p and consequently the circle ABP has equation

$$-a^2pz^2 + a^2(1-p)yz + (b^2 - a^2p)zx + c^2xy = 0. \quad (3.2)$$

The equations of the circles BCQ, CAR may now be written down by cyclic change of x, y, z and a, b, c. Circles ABP and BCQ meet at the point E with co-ordinates (x, y, z), where

$$\begin{aligned} x &= a^2p(c^2p + b^2q(1-p)), \\ y &= b^2pq(a^2p - b^2(1-q)), \\ z &= b^2q(c^2p + b^2q(1-p)). \end{aligned} \quad (3.3)$$

If we now put in the condition that circle CAR also passes through E we get an expression (equal to zero) that factorizes and discarding one factor that cannot hold because p and q are real, we find the condition for concurrence to be

$$r(1-q)/a^2 + p(1-r)/b^2 + q(1-p)/c^2 = 0. \quad (3.4)$$

We now consider the circles ABQ, BCR, CAP. The equation of circle ABQ is

$$-b^2(1-q)z^2 + (a^2 + b^2(1-q))yz + b^2qzx + c^2xy = 0. \quad (3.5)$$

The equations of circles BCR and CAP may now be written down by cyclic change of x, y, z and a, b, c. Circles ABQ and BCR meet at the point F with co-ordinates (x, y, z), where

$$\begin{aligned} x &= b^2(1-q)(a^2(1-r) + b^2r(1-q)), \\ y &= b^2(1-q)(1-r)(c^2(1-r) - b^2q)), \\ z &= c^2(1-r)(a^2(1-r) + b^2r(1-q)). \end{aligned} \quad (3.6)$$

The condition that F also lies on circle CAP, after similar algebra as before, is again found to be Equation (3.4), which completes the proof.

When properties (i) and (ii) are taken into account it follows that if ABC and PQR are in double circular perspective then both ABC and PQR, and PQR and ABC are in triple circular perspective.

It is also the case in projective geometry that if ABC and PQR are two triangles and X, Y are two distinct points then if conics BCPXY, CAQXY, ABRXY have a common point Z then we may say that triangles ABC and PQR are in *conical perspective* by means of triangle XYZ. Generalizations of properties (i) and (ii) now hold by relating X and Y to the circular points at infinity.

4. An example of triple circular perspective

Consider the orthocentroidal circle on GH as diameter, where H is the orthocentre of ABC and G is its centroid. Its equation, see Bradley and Smith [1], is

$$(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2 - a^2yz - b^2zx - c^2xy = 0. \qquad (4.1)$$

The median AG, with equation y = z, meets this circle at a point we call aH and its co-ordinates are easily calculated to be aH(a^2, $b^2 + c^2 - a^2$, $b^2 + c^2 - a^2$). The points bH and cH are similarly

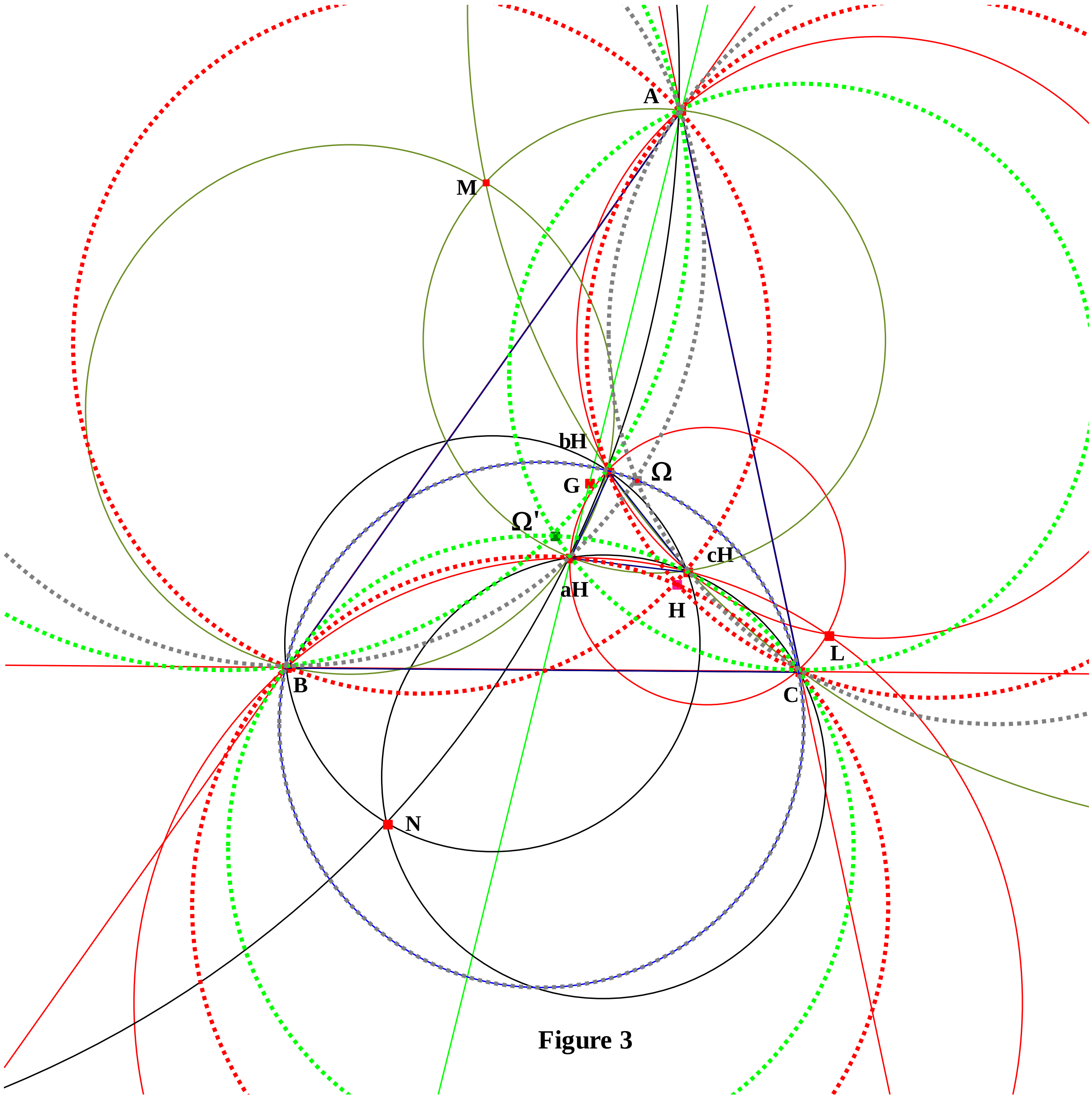

Figure 3

defined and have co-ordinates bH($c^2 + a^2 - b^2$, b^2, $c^2 + a^2 - b^2$) and cH($a^2 + b^2 - c^2$, $a^2 + b^2 - c^2$, c^2). What we show is the following:

Triangles ABC and aHbHcH are in triple circular perspective.

The method of proof is as follows:

(i) We show that circles ABaH, BCbH, CAcH all pass through the Brocard point Ω with co-ordinates $(1/b^2, 1/c^2, 1/a^2)$.

(ii) We then show that circles ABbH, BCcH, CAaH all pass through the Brocard point Ω' with co-ordinates $(1/c^2, 1/a^2, 1/b^2)$.

(iii) It then follows by property (ii) that the circles ABcH, BCaH, CAbH all pass through a fixed point, which we show is the orthocentre H.

(iv) It then follows by property (i) that triangles ABC and aHbHcH are in triple circular perspective.

(v) As a corollary it then follows that circles BaHcH, CbHaH, AcHbH have a common point L, circles BaHbH, CbHcH, AcHaH have a common point M and circles BcHbH, CaHcH, AbHaH have a common point N.

These results are illustrated in Figure 3.

Proof of (i)
The equation of the circle BCbH is

$$b^2x^2 + (b^2 - c^2)xy - a^2yz = 0. \tag{4.2}$$

It is easily checked that this circle passes through $\Omega(1/b^2, 1/c^2, 1/a^2)$. The equations of circles CAcH, ABaH may be written down from Equation (4.2) by cyclic change of x, y, z and a, b, c. And it may now be checked that both these circles also pass through Ω.

Proof of (ii)
The equation of the circle BCcH is

$$c^2x^2 - a^2yz + (c^2 - b^2)zx = 0. \tag{4.3}$$

It is easily checked that this circle passes through $\Omega'(1/c^2, 1/a^2, 1/b^2)$. The equations of circles CAaH, ABbH may be written down from Equation (4.3) by cyclic change of x, y, z and a, b, c. And it may now be checked that both these circles also pass through Ω'.

Proof of (iii)
It now follows from Section 3 that the circles ABcH, BCaH, CAbH all pass through a fixed point, but it remains to show that fixed point is H. In fact the equation of circle BCaH is

$$a^2yz + b^2zx + c^2xy - (b^2 + c^2 - a^2)x(x + y + z) = 0. \tag{4.4}$$

It may now be checked that this circle passes through $H(1/(b^2 + c^2 - a^2), 1/(c^2 + a^2 - b^2), 1/(a^2 + b^2 - c^2))$.

Proof of (iv)
It now follows by Section 2 that triangles ABC and aHbHcH are in triple circular perspective.

Proof of (v)
As a corollary it follows immediately from (iv) that circles BaHcH, CbHaH, AcHbH have a common point L, circles BaHbH, CbHcH, AcHaH have a common point M and circles BcHbH, CaHcH, AbHaH have a common point N.

We do not find the co-ordinates of L, M, N, as it is really only their existence which is significant. In any case experience suggests that the algebra computer package *DERIVE*, which we use, would not be able to find them, the solving of algebraic (rather than numerical) simultaneous quadratics not being possible unless they are very straightforward as in Section 3.

It may just be added that there is a great deal more to this configuration, once the positions of the centres of the eighteen circles have been identified. Many similarities exist, but that is another story.

Dedication
This article is dedicated to my great nephew Matthew Joshua Bradley, it having been composed during the first two weeks of his life.

Reference
1. C.J.Bradley and G.C.Smith, *The locations of triangle centres*, Forum. Geom., 6 (2006) 57-70.

Flat 4,
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP.