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The Midpoint Conic

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Abstract: It is familiar that the feet of a pair of Cevians are co-conic. We prove here that their midpoints are also co-conic.

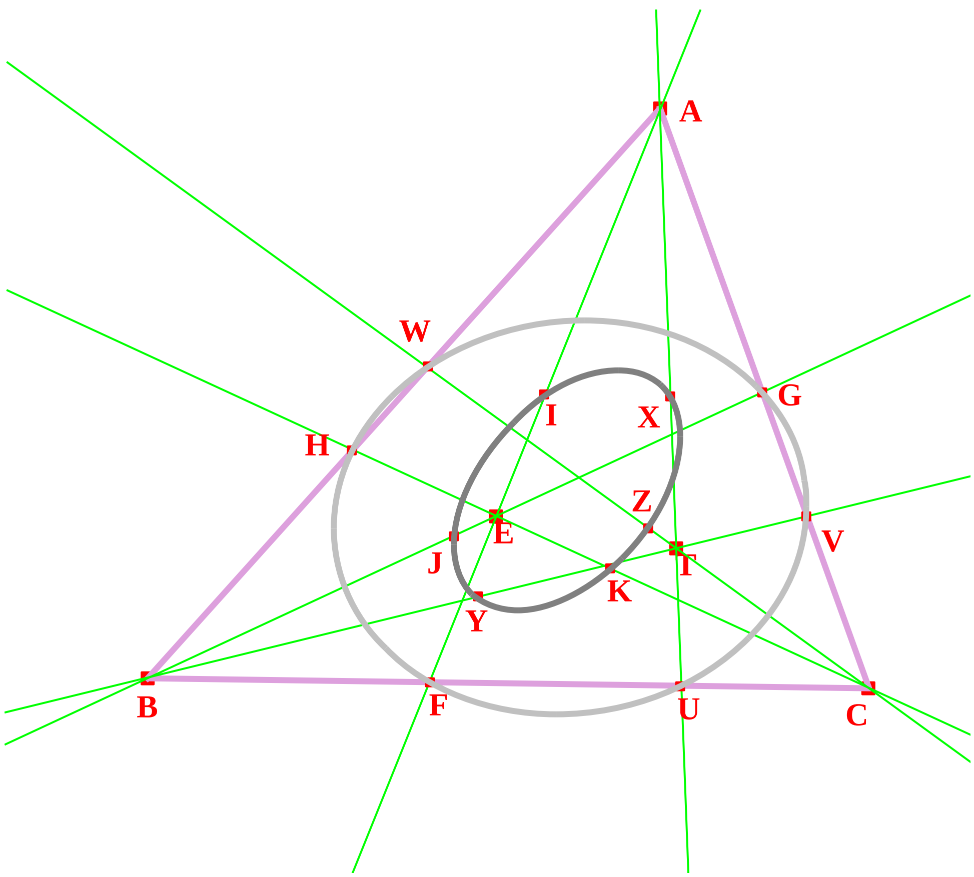

Fig. 1
The midpoints of the Cevians lie on a conic

1. The Cevians

We take the Cevian points E and T to have co-ordinates E(f, g, h) and T(u, v, w). The feet of the Cevians through E therefore have co-ordinates F(0, g/(g + h), h/(g + h)), G(f/(h + f), 0, h/(h + f)) and H(f/(f + g), g/(f + g), 0). And the feet of the Cevians through T have co-ordinates U(0, v/(v + w), w/(v + w)), V(u/(w + u), 0, w/(w + u)), W(u/(u + v), v/(u + v), 0).

2. The conic FGHUVW

This conic has an equation of the form

$$ax^2 + by^2 + cz^2 + 2pyz + 2qzx + 2rxy = 0, \qquad (2.1)$$

where we find by substitution of co-ordinates that

$a = 2ghvw$, $b = 2hfwu$, $c = 2fguv$, $p = – fu(gw + hv)$, $q = – gv(hu + fw)$, $r = hw(fv + gu)$. (2.2)

3. The midpoints IJKXYZ of the Cevians

The co-ordinates of these points are as follows:

I, the midpoint of AF: $x = ½$, $y = g/\{2(g + h)\}$, $z = h/\{2(g + h)\}$, (3.1)

J, the midpoint of BG: $x = f/\{2(h + f)\}$, $y = ½$, $z = h/\{2(h + f)\}$, (3.2)

K, the midpoint of CH: $x = f/\{2(f + g)\}$, $y = g/\{2(f + g)\}$, $z = ½$, (3.3)

X, the midpoint of AU: $x = ½$, $y = v/\{2(v + w)\}$, $z = w/\{2(v + w)\}$, (3.4)

Y, the midpoint of BV: $x = u/\{2(w + u)\}$, $y = ½$, $z = w/\{2(w + u)\}$, (3.5)

Z, the midpoint of CW: $x = u/\{2(u + v)\}$, $y = v/\{2(v + u)\}$, $z = ½$. (3.6)

4. The conic IJKXYZ

This has an equation of the form (2.1), where we find

$$a = f(u + v + w) + g(u + v – w) + h(u – v + w), \quad (4.1)$$

$$b = g(u + v + w) + h(v + w – u) + f(v – w + u), \quad (4.2)$$

$$c = h(u + v + w) + f(w + u – v) + g(w – u + v), \quad (4.3)$$

$$p = fu – (g + h)(v + w), \quad (4.4)$$

$$q = gv – (h + f)(w + u), \quad (4.5)$$

$$r = hw – (f + g)(u + v). \quad (4.6)$$

There is no conic through the quarter way down points of the Cevians and there appears to be no obvious geometrical link between the conics in Sections 2 and 4 (such as a relationship of their centres and the Cevian points).

It should also be noted that a midpoint conic also exists when the six points FGHUVW are the intersections of any conic with the sides of a triangle. The proof of which is left to the reader.

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