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Cevian Perspectivity

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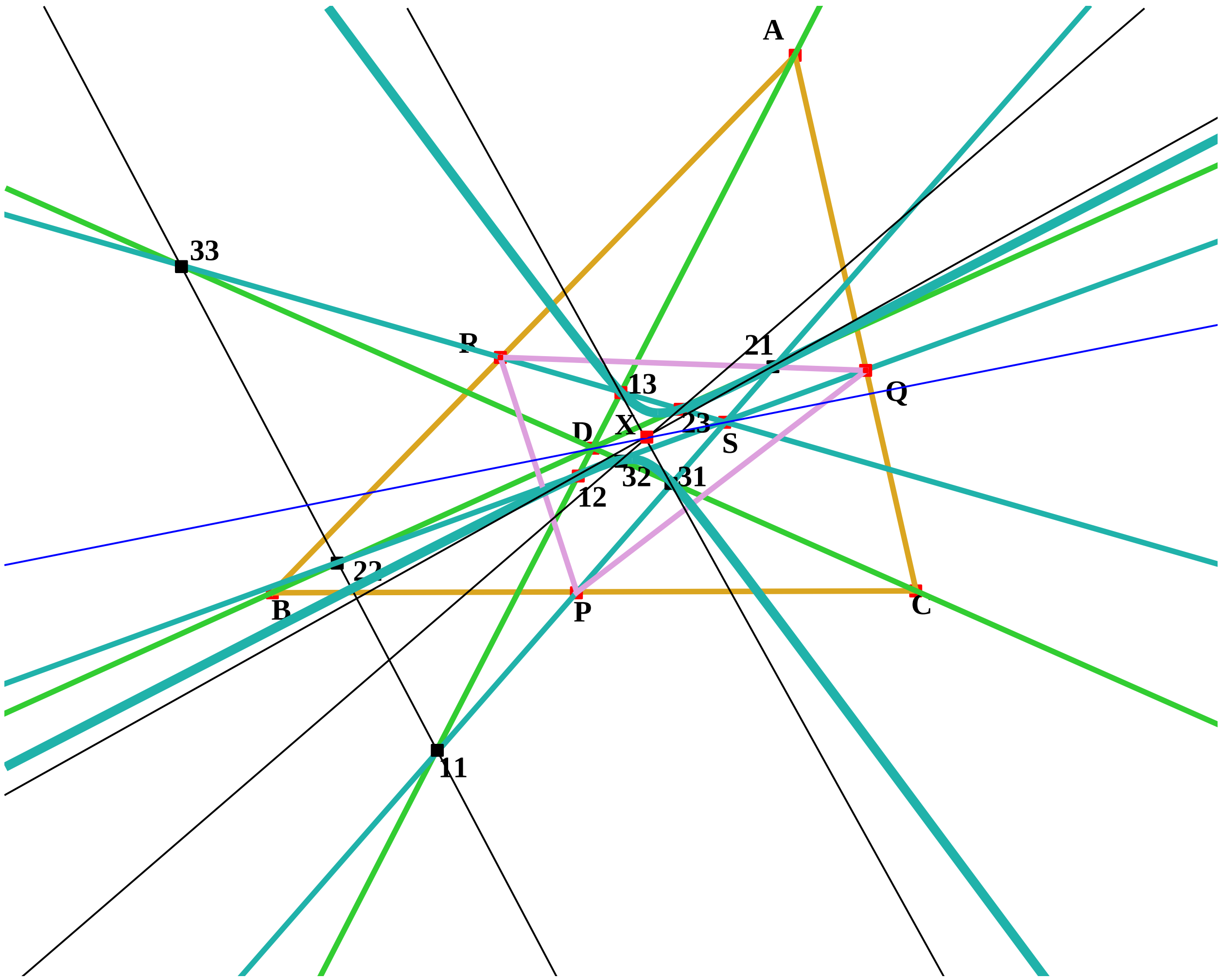

Fig. 1

Abstract: Triangle ABC and PQR with Cevian points D and S respectively are said to be in Cevian perspective if the points 11 = AD^PS, 22 = BD^QS, 33 = CD^RS are collinear. If points are labelled as follows: 12 = AD^QS,13 = AD^RS, 21 = BD^PS, 23 = BD^RS, 31= CD^PS, 32 = CD^QS, then the following results hold: (i) triangles 13 23 21 and 31 32 12 are in (ordinary) perspective through a vertex X, (ii) the points 13 23 21 31 32 12 are co-conic and (iii) X lies on the line SD.

1. Introduction

Projective co-ordinates are used with ABC the triangle of reference and S the unit point. Whilst (i) and (ii) are easily established by pure methods (iii) is the interesting feature of Cevian perspectivity and for this an analytic proof is required. For ease of working we suppose that P, Q, R lie on BC, CA, AB respectively, but the theorems are true under more general conditions, as *CABRI II plus* confirms.

2. The main points and lines

We suppose D has the co-ordinates D(a, b, c) (not the sides of a triangle). Also let P(0, u, 1 – u), Q(1 – v, 0, v), R(w, 1 – w, 0) and S(1, 1, 1). Lines AD, BD, CD have equations bz = cy, cx = az, ay = bx respectively. Lines PS, QS, RS have equations as follows:

PS: $(1 - 2u)x + (u - 1)y + uz = 0,$ (2.1)

QS: $vx + (1 - 2v)y + (v - 1)z = 0,$ (2.2)

RS: $(w - 1)x + wy + (1 - 2w)z = 0.$ (2.3)

The co-ordinates of the points 11 = AD^PS 22 = BD^QS, 33 = CD^RS may now be obtained and are:

11: $x = uc + (u - 1)b, y = (2u - 1)b, z = (2u - 1)c,$ (2.4)

22: $x = (2v - 1)a, y = va + (v - 1)c, z = (2v - 1)c,$ (2.5)

33: $x = (2w - 1)a, y = (2w - 1)b, z = wb + (w - 1)a.$ (2.6)

The condition for 11 22 33 to be collinear, so that a Cevian perspectivity exists, is given by the equation

$$w = (1/k)(a^2(b(u(3v - 2) - v + 1) + cv(u - 1)) - ac(b(u(9v - 5) - 5v + 3) + c(1 - v)(u - 1)) + bc(2v - 1)(b(u - 1) + cu)), \quad (2.7)$$

where

$$k = a^2(b(u(3v - 2) - v + 1) + cv(3u - 2)) + a(b^2(u(3v - 2) - v + 1) - bc(9u(2v - 1) - 9v + 5) + c^2(v - 1)(3u - 2)) + bc(3v - 1)(b(u - 1) + cu). \quad (2.8)$$

It is now possible to obtain the co-ordinates of the six main points, which are

12 = AD^QS: $x = b(2v - 1)) + c(1 - v), y = bv, z = cv,$ (2.9)

23 = BD^RS: $x = aw, y = a(1 - w) + c(2w - 1), z = cw,$ (2.10)

31 = CD^PS: $x = au, y = bu, z = b(1 - u) + a(2u - 1),$ (2.11)

21 = BD^PS: $x = a(u - 1), y = a(2u - 1) - cu, z = c(u - 1),$ (2.12)

32 = CD^QS: $x = a(v -0 1), y = b(v - 1), z = b(2v - 1) - av,$ (2.13)

13 = AD^RS: $x = c(2w - 1) - bw, y = b(w - 1), z = c(w - 1).$ (2.14)

3. The conic and the condition it passes through all six main points

The equation of a conic is

$$px^2 + qy^2 + rz^2 + 2fyz + 2gzx + 2hxy = 0, \quad (3.1)$$

and the values of the constants p, q, r, f, g, h so that it passes through 12, 23, 31, 21, 32 are

$$p = 2cv(a^3v(w - 1)(2u - 1) + a^2(1 - 2u)(2b(w - 1)(2v - 1) + c(vw + w - 1)) - a(b^2(u(v(w + 2) - w - 1) + v(w - 2) + 1) - bc(u(v(19w - 10) - 9w + 5) + 2(1 - 2w)(2v - 1)) + c^2(v - 1)(2w - 1)(2u - 1)) - bc(b(u - 1) + cu)(v(3w - 2) - w + 1)), \quad (3.2)$$

$$q = 2acw(1 - u)(av + b(1 - 2v) + c(v - 1))^2, \quad (3.3)$$

$$r = 2acu(1 - v)(av + b(1 - 2v) + c(v - 1))(a(w - 1) + bw + c(1 - 2w)), \quad (3.4)$$

$$f = -a(a^2b(u(3v - 2) - v + 1)(v(3w - 2) - w + 1) - c(u(v^2(7w - 5) + 2v(3 - 4w) + 2(w - 1)) + (1 - v)(v(2w - 1) - w + 1))) + ac(c(u(2v^2(5w - 3) + v(8 - 11w) + 3(w - 1)) + 2(1 - w)(v - 1))) + c(b^2w(2v - 1)(uv + v - 1) + bc(v - 1)(u(v(w - 2) - w + 1) + v(2 - 5w) + 3w - 1) - c^2(v - 1)^2(u(w - 1) - 2w + 1))), \quad (3.5)$$

$$g = a^3cv(w - 1)(u(v - 2) - v + 1) + a^2(b^2(u(3v - 2) - v + 1)(v(3w - 2) - w + 1) - bc(u(v^2(7w - 6) + 11v(1 - w) + 4(w - 1)) + 2(1 - w)(2v^2 - 3v + 1)) - c^2(u(v - 2) - v + 1)(vw + w - 1)) - ac(b^2(u(v^2(17w - 10) + v(11 - 19w) + 5w - 3) + v^2(2 - 5w) + v(7w - 3) - 2w + 1) + bc(1 - 2v)(u(v(7w - 4) - 9w + 5) + 2(1 - 2w)(v - 1)) + c^2(v - 1)(2w - 1)(u(v - 2) - v + 1)) + bc^2u(b(2v - 1) + c(1 - v))(v(3w - 2) - w + 1), \quad (3.6)$$

$$h = c(a^3v^2(u(w + 1) - 1) + a^2(b(u(5v - 2) - 3v + 1)(v(w - 2) - w + 1) - c(2u(v^2(2w - 3) + v(4 - 3w) + w - 1) + 2v^2 + v(3w - 4) - w + 1)) + a(b^2(4v^2 - 4v + 1)(u(w + 1) - 1) + bc(1 - 2v)(u(v(11w - 4) - 5w + 1) - 2(v(3w - 1) - w)) + c^2(u(v^2(13w - 7) + v(8 - 15w) + 2(2w - 1)) + (1 - 2w)(3v^2 - 4v + 1))) + bc(u - 1)(b(2v - 1) + c(1 - v))(v(3w - 2) - w + 1)). \quad (3.7)$$

The condition (2.7) now ensures that this conic also passes through the point 13. It is worth mentioning the fact that three general lines intersect three other general lines in nine points and three of the nine points such as 11, 22, 33 are collinear, then except in special circumstances, the other six points lie on a conic (or two other straight lines). This is a general theorem in quantitative geometry (see, for example, *Sylvester's Geometry Ancient and Modern*).

4. The point X

The equation of the line SD is

$$(b - c)x + (c - a)y + (a - b)z = 0. \quad (4.1)$$

In view of the line 11 22 33 (regarded as a Desargues' axis it follows that triangles 13 23 21 and 31 32 12 are in perspective with a vertex X and it is found that the co-ordinates of X are

$$x = (a^2(b(u(3v - 2) - v + 1) - c(u(v - 2) + 1)) + 2ac(1 - 2u)(b(2v - 1) + c(1 - v)) + c(b^2(u - 1)(2v - 1) + bc(uv + v - 1) + c^2u(1 - v))), \quad (4.2)$$

$$y = a^2cv(2u - 1) + a(b^2(u(3v - 2) - v + 1) + 2bc(1 - 2v)(2u - 1) + c^2(u(v - 2) - v + 1)) - c(b^2(u(v - 1) + v) + 2bcu(1 - 2v) + c^2u(v - 1)), \quad (4.3)$$

$$z = c(a^2v(2u - 1) - 2av(b(u - 1) + cu) + b^2(u - 1)(2v - 1) + 2bc(1 - v)(u - 1) + c^2(u(2v - 1) - v + 1)). \quad (4.4)$$

It is now straightforward to show that X lies on SD, which is a slightly surprising result. The more general situation is shown in Fig. 2 below.

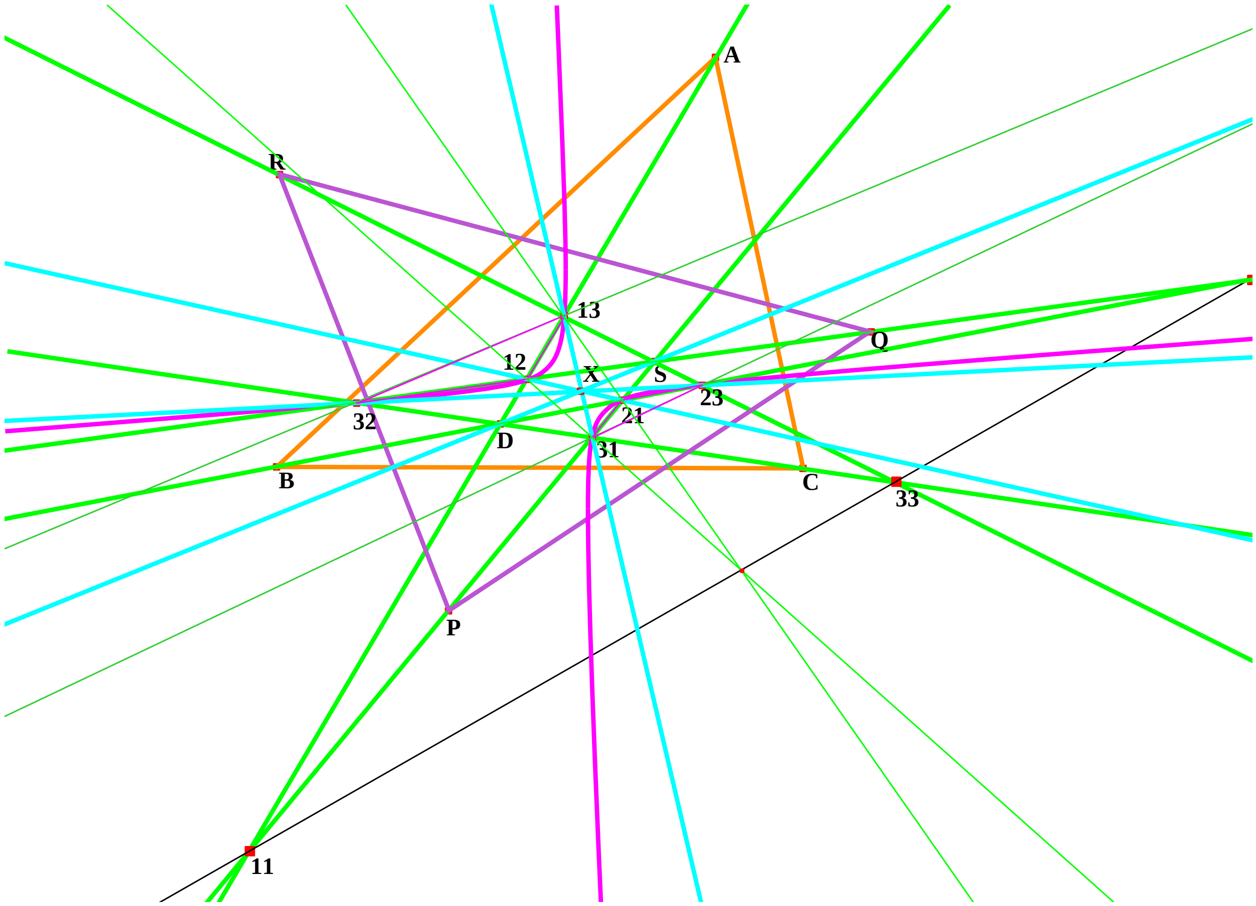

Fig. 2

Another point of interest is that as a result of the perspective of the two triangles that line 32 13 intersects 23 31 at a point on line 11 22 33 as do lines 12 31 and 13 21.

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