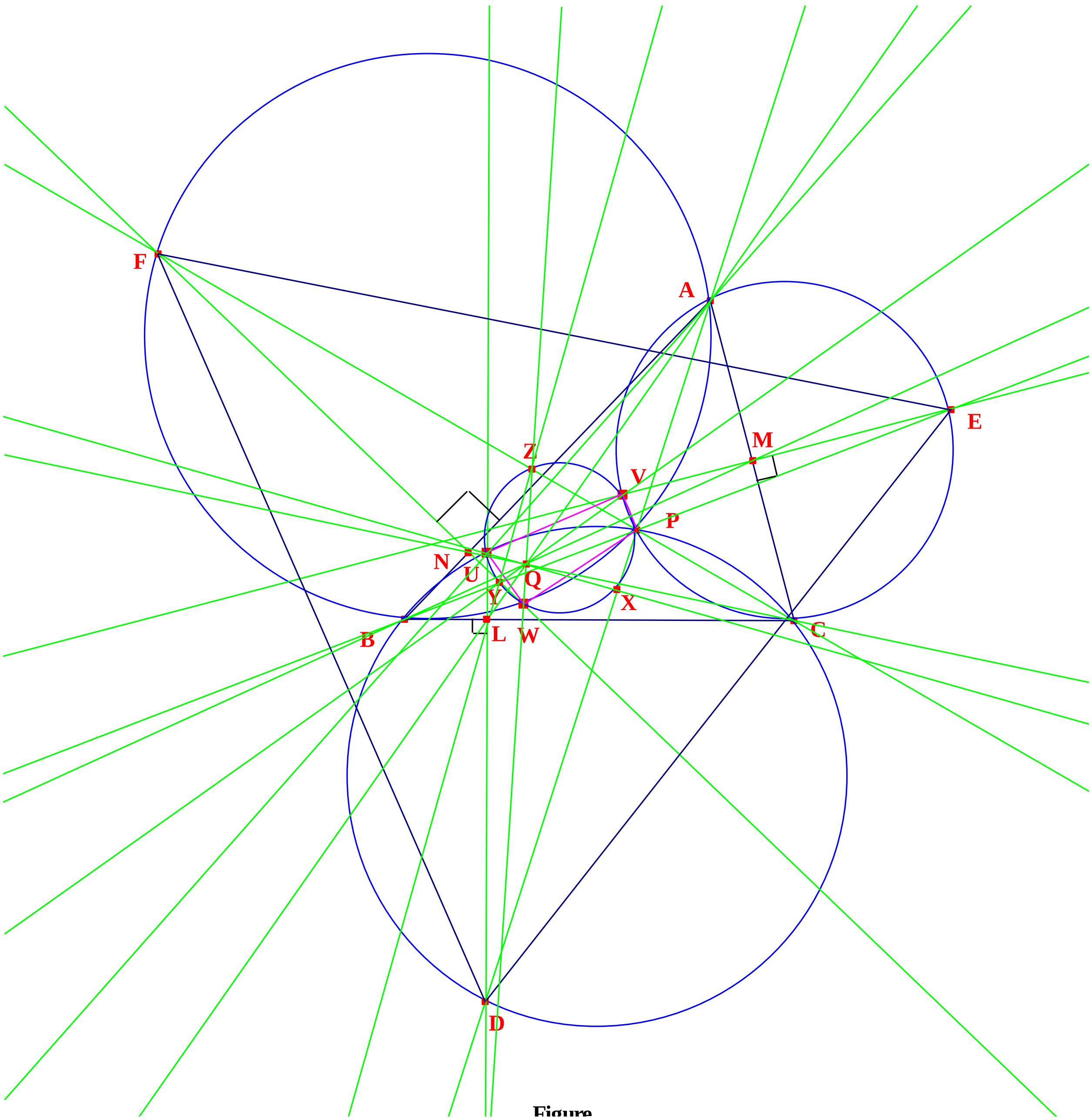
Article 18

Some Special Circles in a Triangle

Christopher J Bradley



Figure

1. Introduction

We study a construction in a triangle ABC that assigns to any point P, not on the sides, circumcircle or altitudes of a triangle, a unique circle passing through P. This is done by finding three points U, V, W that define the circle and once it has been shown that U, V, W, P are concyclic, then four other points Q, X, Y, Z are determined with the property that UX, VY, WZ are concurrent at Q.

The construction is as follows. Choose P and draw the circles BPC, CPA, APB, which we denote by Γ_1, Γ_2, Γ_3 respectively. Now draw AP, BP, CP to meet Γ_1, Γ_2, Γ_3 respectively at D, E, F. Draw through D the perpendicular to BC to meet Γ_1 again at U, with V, W defined similarly on perpendiculars to CA, AB respectively. Then it turns out that U, V, W, P are concyclic and lie on a circle we denote by Γ_P. Now draw AP, BP, CP to meet Γ_P at X, Y, Z respectively. Then it may be proved that UX, VY, WZ are concurrent at a point Q. Furthermore AQ, BQ, CQ intersect BC, CA, AB respectively at points L, M, N that also lie respectively on DU, EV, FW. Those familiar with the construction of Hagge circles, see Bradley and Smith [1], will recognize the similarity that exists between the two constructions, except that Hagge circles define all circles passing through the orthocentre H, whereas our construction gives one circle (for any given triangle) passing through all points P not lying on the sides or altitudes of ABC. See the figure above for an illustration of the construction.

In the analysis that follows we take P to be the centroid G, so all that is proved is that the construction works in this case. However, the computer geometry software package *CABRI II plus* is so accurate that we may be sure that the construction holds generally. However, the algebra involved for a general point is so formidable (even with the aid of a computer algebra software package) that we offer this as a most potent excuse for not covering the general case. We use areal co-ordinates throughout, an account of which is given by Bradley [2, 3].

2. The circles BGC, CGA, AGB and the points D, E, F

The equation of a circle using areal co-ordinates is always of the form

$$a^2yz + b^2zx + c^2xy + (ux + vy + wz)(x + y + z) = 0, \qquad (2.1)$$

where u, v, w are constants to be determined and a, b, c are the side lengths of ABC. To find the equation of circle BGC we put the co-ordinates of points B, G, C in Equation (2.1) and get three equations to determine u, v, w. The values obtained are $u = -\frac{a^2+b^2+c^2}{3}, v = 0, w = 0$ and the equation of circle BGC is accordingly

$$(a^2 + b^2 + c^2)x^2 - 3a^2yz + (c^2 + a^2 - 2b^2)zx + (a^2 + b^2 - 2c^2)xy = 0. \qquad (2.2)$$

The equations of circles CGA, AGB may be obtained by cyclic change of x, y, z and a, b, c. The equation of AG is y = z and this meets the circle BGC at the point D with co-ordinates D($-3a^2$, ($a^2 + b^2 + c^2$), ($a^2 + b^2 + c^2$)). Points E, F have co-ordinates that may be obtained from those of D by cyclic change of x, y, z and a, b, c.

3. Lines through D, E, F perpendicular to BC, CA, AB

Finding perpendicular lines when using areal co-ordinates is tiresome, but the results are known, see [3], and may therefore be quoted.

If we take a point T with co-ordinates (d, e, f) then the foot of the perpendicular from T on the line BC has co-ordinates $\left(0, \frac{2a^2e+(a^2+b^2-c^2)}{2a^2}, \frac{2a^2f+(c^2+a^2-b^2)}{2a^2}\right)$ and consequently the equation of the line perpendicular to BC, x = 0, through the point T has equation

$$\big(a^2(e-f)-(b^2-c^2)(e+f)\big)x+\big((b^2-c^2)d-a^2(d+2f)\big)y+(a^2(d+2e)+(b^2-c^2)d)z=0. \quad (3.1)$$

If we now put d = $-3a^2$, e = f = ($a^2 + b^2 + c^2$), we get the equation of DU, which is

$$2(a^2+b^2+c^2)(c^2-b^2)x+a^2(c^2+a^2-5b^2)y-a^2(a^2+b^2-5c^2)z=0. \quad (3.2)$$

The equations of EV and FW may be written down from Equation (3.2) by cyclic change of x, y, z and a, b, c.

4. Points U, V, W and the circle through G

The line DU, with Equation (3.2), meets Γ_1 with Equation (2.2) at the point U with co-ordinates (x, y, z), where

$$x = a^2(a^2+b^2-5c^2)(c^2+a^2-5b^2),$$
$$y = a^6-8a^4c^2+a^2(3b^4-4b^2c^2+17c^4)+2(b^2-c^2)(2b^4-11b^2c^2+5c^4), \quad (4.1)$$
$$z = (c^2+a^2-5b^2)(a^4-a^2(3b^2+c^2)+2(b^2-c^2)(b^2-2c^2)).$$

The co-ordinates of the points V, W may be found by cyclic change of x, y, z and a, b, c.

We now substitute the co-ordinates of U, V, W into Equation (2.1) to obtain three equations for u, v, w. These values are then substituted back in Equation (2.1) to obtain the unpromising equation of the circle Γ_G, which is

$$(a^2+b^2+c^2)(b^2+c^2-5a^2)(2b^2+2c^2-a^2)x^2+\ \dots+\cdots$$
$$-(23a^6-51a^4(b^2+c^2)+9a^2(3b^4-2b^2c^2+3c^4)-(b^2+c^2)(7b^4-22b^2c^2+7c^4))yz\ -\cdots-\cdots=0. \quad (4.2)$$

Terms in y^2, z^2, and zx, xy may be obtained by cyclic change of a, b, c of those in x^2 and yz.
It may now be checked that this circle passes through G(1, 1, 1).

5. The points X, Y, Z and the linking point Q

The equation of AG is y = z and this meets the circle Γ_G with Equation (4.2) at the point X with co-ordinates (x, y, z), where

$$x = -19a^4 + 10a^2(b^2 + c^2) - 7b^4 + 22b^2c^2 - 7c^4,$$
$$y = (a^2 + b^2 + c^2)(5a^2 - b^2 - c^2), \tag{5.1}$$
$$z = (a^2 + b^2 + c^2)(5a^2 - b^2 - c^2).$$

The co-ordinates of Y, Z, where BG, CG respectively meet Γ_G may be obtained from those of X by cyclic change of x, y, z and a, b, c.

The equations of the lines UX, VY, WZ are lengthy and no good purpose would be served by writing them down. However, the algebra computer package *DERIVE*, which we used throughout, showed that the three lines meet at the point Q, whose co-ordinates (x, y, z) are given by

$$x = \frac{1}{5a^2 - b^2 - c^2}, y = \frac{1}{5b^2 - c^2 - a^2}, z = \frac{1}{5c^2 - a^2 - b^2}. \tag{5.2}$$

The apparent simplicity of this result is very gratifying. We use the term *linking point* for Q, as it links the point G with the points lying on circle Γ_G. This it does in two ways. The first has just been explained as Q is the point of concurrency of UX, VY, WZ. The second way is that the lines AQ, BQ, CQ meet the lines DU, EV, FW respectively at points L, M, N lying on BC, CA, AB respectively. To see this, note that the equation of AQ is

$$(5b^2 - c^2 - a^2)y = (5c^2 - a^2 - b^2)z, \tag{5.3}$$

and from Equation (3.2) it can be seen that DU meets BC at this point also. Note that Q may lie outside triangle ABC.

References

1. C. J Bradley & G.C.Smith, Math. Gaz. pp202 -207 July 2007.
2. C. J. Bradley, *Challenges in Geometry*, Oxford (2005).
3. C. J .Bradley, *The Algebra of Geometry*, Highperception, Bath (2007).

Flat 4,
Terrill Court,
12/14 Apsley Road,
BRISTOL BS8 2SP