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Special Pascal Lines

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Abstract: By considering the hexagon as two triangles in perspective and labelling their vertices as well as those of the defining conic, it is shown that two of the Pascal lines pass through points that are points of concurrence of three lines.

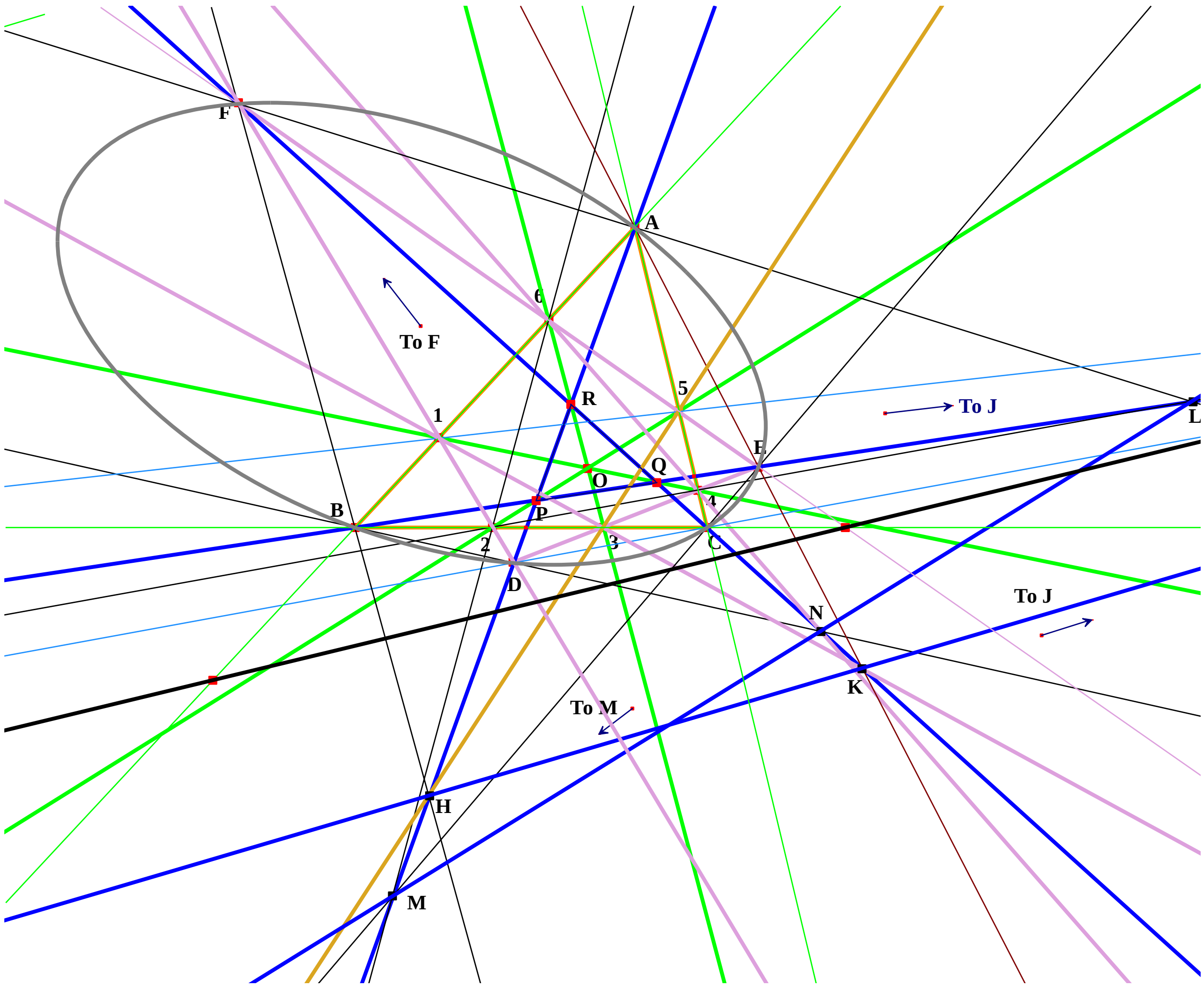

Fig. 1
The two special Pascal lines LMN and HJK

1. Introduction

The starting point in the construction of Fig.1 is to draw triangles 123 and 456 in perspective through a point O. The triangle ABC is formed by the sides 61 54 32 and the triangle DEF is formed from the sides 12 34 56. The six points A, B, C, D, E, F now lie on a conic (a proof of which is given late in the article). We use homogeneous projective co-ordinates with 123 as triangle of reference and the vertex O of perspective as unit point O (1, 1, 1). We find six points L = BE^AF^24, M = CE^AD^26, N = BD^CF^46, H = AD^BF^35, J = BE^CD^15, K = AE^CF^13 and we prove LMN and HJK are Pascal lines, which we call *special* because they are points of concurrence of three lines. There are other points of concurrence of three lines which also define Pascal lines, and have a certain interest.

2. The co-ordinates of points and equations of sides and joins of triangles 123 and 456

With 123 of triangles of reference we have co-ordinates 1(1, 0, 0), 2(0, 1, 0), 3(0, 0, 1) and we take O, the vertex of perspective of the two triangles as O(1, 1, 1). It is the sufficiently general to take 4(1, a, a), 5(b, 1, b), 6(c, c, 1), where a, b, c are arbitrary constants. Equations of sides of the two triangles are now:

23: $x = 0$, 31: $y = 0$, 12: $z = 0$, 14: $y = z$, 25: $z = x$, 36: $x = y$, 34: $y = ax$, 61: $y = cz$, 35: $x = by$, 15: $z = by$, 24: $z = ax$, 26: $x = cz$,

45: $a(b - 1)x + b(a - 1)y + (1 - ab)z = 0,$

56: $(bc - 1)x + b(1 - c)y + c(1 - b)z = 0,$ (2.1)

64: $a(c - 1)x + (1 - ca)y + c(a - 1)z = 0.$

3. The co-ordinates of points and equations of joins of triangles ABC and DEF

The points A, B, C, D, E, F are now the intersections of lines obtained in Section 2 and their co-ordinates are

A = 61^45: $x = ab(c - 1) - bc + 1$, $y = ca(1 - b)$, $z = a(1 - b)$. (3.1)

B = 61 23: $x = 0$, $y = c$, $z = 1$. (3.2)

C = 23^45: $x = 0$, $y = ab - 1$, $z = b(a - 1)$. (3.3)

D = 12^34: $x = 1$, $y = a$, $z = 0$. (3.4)

E = 34^56: $x = c(1 - b)$, $y = ca(1 - b)$, $z = ab(c - 1) - bc + 1$. (3.5)

F = 12^56: $x = b(c - 1)$, $y = bc - 1$, $z = 0$. (3.6)

The equations of the joins of these two triangles are

AD: $a(b - 1)x + (1 - b)y + (c - 1)(ab - 1)z = 0.$ (3.7)

AE: $ca(a - 1)(b - 1)(bc - 1)x + (a^2b^2(c - 1)^2 - a(b^2c(2c - 1) + 2b(1 - 2c) + c) + b^2c^2 - 2bc + 1)y + ca(b - 1)(c - 1)(ab - 1)z = 0.$ (3.8)

AF: $a(b - 1)(bc - 1)x + ab(1 - c)(b - 1)y + (ab(c - 1)(2bc - c - 1) - (bc - 1)^2)z = 0.$ (3.9)

BD: $ax - y + cz = 0.$ (3.10)

BE: $(a - 1)(bc - 1) + (1 - b)(y - cz) = 0.$ (3.11)

BF: $(bc - 1)x + b(1 - c)(y - cz) = 0.$ (3.12)

CD: $ab(1 - a)x + b(a - 1)y + (1 - ab) z = 0.$ (3.13)

CE: $(a^2b(b(2c-1)-c)+2ab(1-bc)+bc-1)x+$
$c(1-b)(b(a-1)y+(1-ab))z=0$ (3.14)

CF: $(a-1)(bc-1)x+(1-c)(b(a-1)y+(1-ab))z=0.$ (3.15)

4. The points P, Q, R

These points are each the concurrencies of three lines and their co-ordinates are as follows.

P = AD^BE^25: $x = b-1,\ y = a(bc-1)-c+1,\ z = b-1.$ (4.1)

Q = BE^CF^14: $x = (1-c)(b-1),\ y = (a-1)(bc-1),\ z = (a-1)(bc-1).$ (4.2)

R = AD^CF^36: $x = (1-c)(ab-1),\ y = (1-c)(ab-1),\ z = (a-1)(b-1).$ (4.3)

Points P, Q, R do not lie on Pascal lines but have their properties in consequence of the perspective.

5. The conic ABCDEF

From the co-ordinates of these points it is now possible to show they lie on a conic – this is, of course, a well known result. The conic has an equation of the form

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0, \quad (5.1)$$

where we find

$$u = 2a(1-b)(1-a)(1-bc), \quad (5.2)$$
$$v = 2b((1-a)(1-b)(1-c), \quad (5.3)$$
$$w = 2c((1-b)(1-c)(1-ab), \quad (5.4)$$
$$f = (1-b)(1-c)(ab(c+1)-bc-1), \quad (5.5)$$
$$g = (1-c)(a-1)(ab(2bc-c-1)-bc+1), \quad (5.6)$$
$$h = (1-a)(1-b)(ab(c-1)+bc-1). \quad (5.7)$$

6. The Special Pascal lines LMN and HJK

There are, of course, a vast number of Pascal lines, but there are only two that have the special property that the points defining them are points of concurrence of three lines in the figure (rather than two). We first give details of the line LMN. Their co-ordinates are

L = AF^BE^24 : $x = b-1,\ y = a(2bc-c-1)-bc+1,\ z = a(b-1).$ (6.1)

M: AD^CE^26: $x = c(b-1),\ y = a(b(2c-1)-c)-c+1,\ z = b-1.$ (6.2)

N: BD^CF^46: $x = 1-c,\ y = a-c,\ z = a-1.$

It may now be checked that L, M, N lie on the Pascal line with equation

$$(a^2(b(2c-1)-c)+2a(1-bc)+bc-1)x+(1-b)(ca-1)y+$$
$$(a(b(2c^2-2c+1)-c^2)-bc^2+2c-1)z=0. \quad (6.3)$$

We now give details of the line HJK. Their co-ordinates are

H = AD^BF^35: $x = b(1 – c)$, $y = 1 – c$, $z = b – 1$. (6.4)

J = BE^CD^15 : $x = 1 – b$, $y = a – 1$, $z = b(a – 1)$. (6.5)

K = AE^CF^13: $x = (1 – c)(ab – 1)$, $y = 0$, $z = (a – 1)(bc – 1)$. (6.6)

It may now be checked that H, J, K lie on the Pascal line with equation

$$(a - 1)(bc - 1)x - (ab^2(c - 1) - b^2c + 2b - 1)y + (1 - c)(1 - ab)z = 0. \qquad (6.7)$$

It is also the case that P, R, M, H lie on AD, that P, Q, L, J lie on BE and Q, R, N, K lie on CE as may be checked from the definition of the six points L, M, N, H, J, K.

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