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The Brocard Conics

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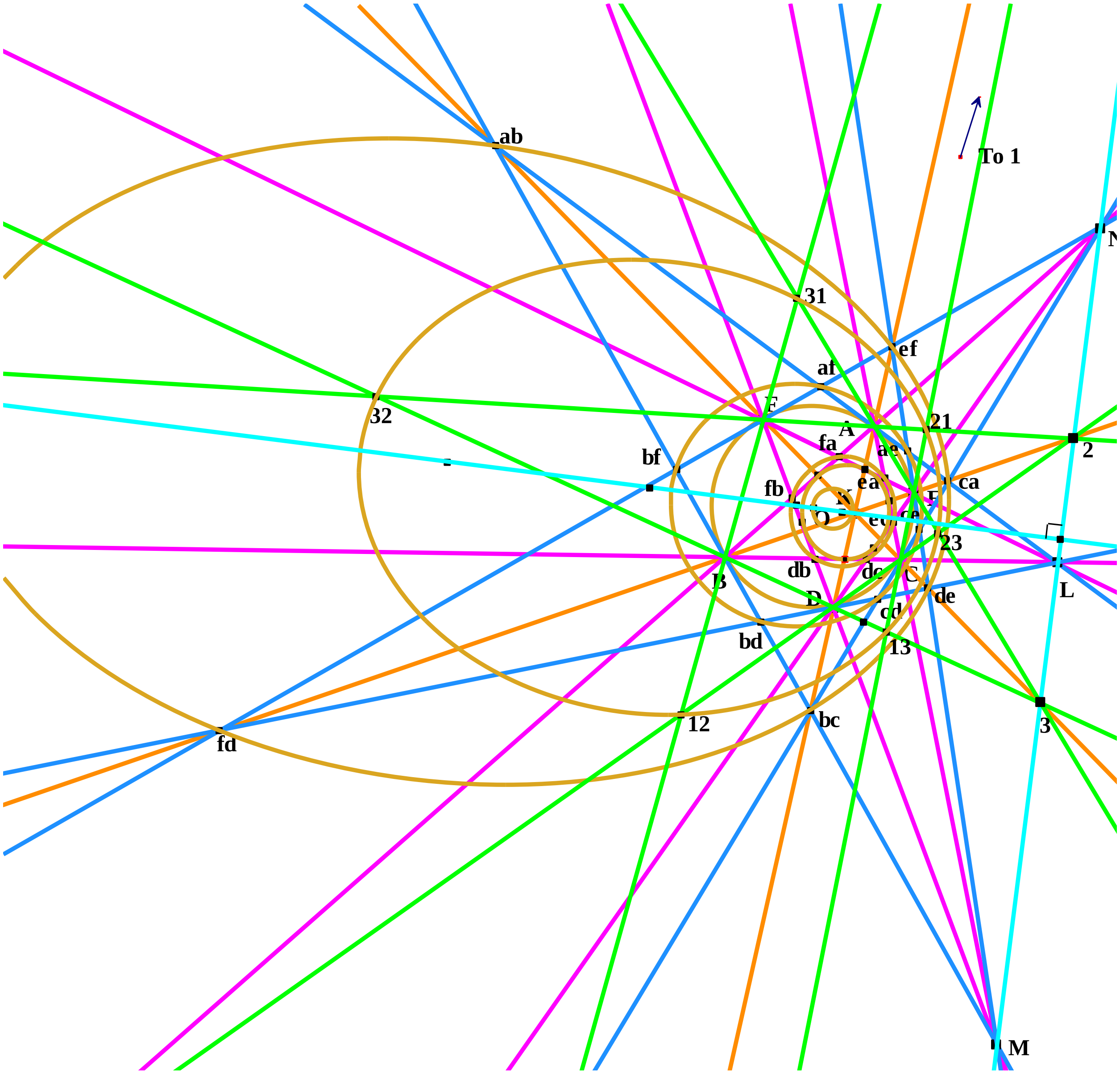

Fig. 1

Abstract: Triangles ABC and DEF are inscribed in a circle and AD, BE, CF are concurrent at the symmedian point K. Sides of the two triangles are extended and the tangents at the vertices are drawn. The polar lines of the two triangles coincide and points of intersection of the sides and tangents are shown to produce in addition four six-point conics all with their centres on the Brocard axis OK, where O is the circumcentre of ABC.

1. Introduction

A triangle ABC, symmedian point K, is inscribed in a circle Σ centre O. Lines AK, BK, CK meet Σ again at D, E, F respectively. It follows that ABC and DEF are a pair of in-perspective triangles in the Brocard porism. Lines EF, BC meet at L, and M and N are similarly defined. The tangents to Σ are drawn at the six vertices and it is proved that the tangents at A and D pass through L. Similarly the tangents at B and E pass through M and the tangents at C and F pass through M.

Point ab is the intersection of the tangents at A and B and point bc, ca, de, ef, fd are similarly defined. We show that these six points lie on a conic. The equations of the lines joining pairs of vertices are obtained. The following points are now obtained: 23 = CD^AE, 31 = AE^BF, 12 = BF^CD, 21 = CE^AF, 32 = AF^BD, 13 = BD^CE. It is then shown that these six points lie on a conic.

It is then proved that AE, BD, FC are concurrent at a point 3 lying on the line LMN. Points 1 and 2 are similarly defined. The point af is defined as the intersection of the tangents at A and F. Points bf, bd, cd, ce, ae are similarly defined. It is the proved that these six points also lie on a conic.

The six points of (internal) intersection of the two triangles are defined as follows: fa = AB^EF, fb = AB^FD, db = BC^FD, de = BC^DE, ec = CA^DE, ea = CA^EF. It is then proved that these six points also lie on a conic.

Finally it is proved that the centres of these conics lie on the Brocard axis OK (the diameter of the 7-point circle). It must be mentioned also that the Brocard ellipse that touches internally all six sides of the two triangles (the inconic of the Brocard porism) also has its centre on OK, so we have a system of five conics altogether all with their centres on a line. See Fig. 1 for a diagram of all these results.

2. Points D, E, F, L, M, N

The equation of the circumcircle Σ is well-known to have the equation

$$a^2yz + b^2zx + c^2xy = 0 \qquad (2.1)$$

Since the symmedian point K has co-ordinates (a^2, b^2, c^2) the line AK has equation $b^2z = c^2y$. The point of intersection D of AK and Σ therefore has co-ordinates $D(-a^2, 2b^2, 2c^2)$. Similarly E and F have co-ordinates $E(2a^2, -b^2, 2c^2)$ and $F(2a^2, 2b^2, -c^2)$.

The equation of EF is

$$b^2c^2x = 2a^2(c^2y + b^2z) \qquad (2.2)$$

The side EF meets BC ($x = 0$) at $L(0, -b^2, c^2)$. Similarly M and N have co-ordinates $M(a^2, 0, -c^2)$ and $N(-a^2, b^2, 0)$. The equation of LMN is thus

$$x/a^2 + y/b^2 + z/c^2 = 0. \qquad (2.3)$$

The tangent to Σ at A has equation

$$c^2y + b^2z = 0 \qquad (2.4)$$

and the tangent to Σ at D has equation

$$4b^2c^2x + c^2a^2y + a^2b^2z = 0. \qquad (2.5)$$

Both these tangents pass through L. Points M and N also lie on corresponding tangents to Σ. It follows that LMN is the common polar line of the two triangles. (In fact LMN has the property that pairs of triangles in the Brocard porism are in triple reverse perspective with vertices on the line LMN.)

Tangents to Σ at B, C and E, F may be written down from equations (2.4) and (2.5) by cyclic change of x, y, z and a, b, c.

3. Points ab, bc, ca, de, ef, fd and the conic on which they lie

Point ab is the intersection of the tangents at A and B. It has co-ordinates $ab(a^2, b^2, -c^2)$. Similarly bc has co-ordinates $bc(-a^2, b^2, c^2)$ and ca has co-ordinates $ca(a^2, -b^2, c^2)$. These are, of course, the three ex-symmedian points of triangle ABC.

Point de is the intersection of tangents to Σ at D and E and has co-ordinates $de(a^2, b^2, -5c^2)$. Similarly ef has co-ordinates $ef(-5a^2, b^2, c^2)$ and fd has co-ordinates $fd(a^2, -5b^2, c^2)$. Since K is also the symmedian point of triangle DEF it follows that these three points are the ex-symmedian points of triangle DEF.

It may now be shown that these six point lie on the conic Σ_1 with equation

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad (3.1)$$

where

$$u = 2b^4c^4,\ v = 2c^4a^4,\ w = 2a^4b^4,\ f = 3a^4b^2c^2,\ g = 3a^2b^4c^2,\ h = 3a^2b^2c^4. \qquad (3.2)$$

4. Sides EF, FD, DE and joins of vertices AE, BF, CD, AF, BD, CE

The equations of these nine lines are:

EF: $$2a^2(c^2y + b^2z) - b^2c^2x = 0. \qquad (4.1)$$

The equations of FD and DE now follow from (4.1) by cyclic change of x, y, z and a, b, c.

AE: $2c^2y + b^2z = 0.$ (4.2)

The equations of BF and CD now follow from (4.2) by cyclic change of x, y, z and a, b, c.

AF $c^2y + 2b^2z = 0.$ (4.3)

The equations of BD and CE now follow from (4.3) by cyclic change of x, y, z and a, b, c.

5. Points 23 = CD^AE, 31 = AE^BF, 12 = BF^CD, 21 = CE^AF, 32 = AF^BD, 13 = BD^CE and the conic on which they lie

These six points have co-ordinates

23: $x = a^2, y = – 2b^2, z = 4c^2.$ (5.1)

The co-ordinates of 31 and 12 follow from equations (5.1) by cyclic change of x, y, z and a, b, c.

21: $x = 4a^2, y = – 2b^2, z = c^2.$ (5.2)

The co-ordinates of 32 and 13 follow from equations (5.2) by cyclic change of x, y, z and a, b, c.

It may now be shown that these six points lie on a conic Σ_2 with equation of the form (3.1) where

$u = 4b^4c^4, v = 4c^4a^4, w = 4a^4b^4, f = 7a^4b^2c^2, g = 7a^2b^4c^2, h = 7a^2b^2c^4.$ (5.3)

6. The points 1, 2, 3

Lines AE and BD meet at the point 3 with co-ordinates $(a^2, b^2, – 2c^2)$. It may be shown that point 3 lies on both LMN and CF. Points 1 = AD^BF^CE and 2 = AF^BE^CD also lie on LMN.

7. Intersections of tangents at the vertices and the conic on which they lie

Point af is the intersection of tangents to Σ at points A and F. It has co-ordinates $(3a^2, b^2, – c^2)$. Points bd, ce have co-ordinates that may be written down from those of af by cyclic change of x, y, z and a, b, c.

Point bf is the intersection of tangents to Σ at points B and F. It has co-ordinates $(a^2, 3b^2, – c^2)$. Points cd, ae have co-ordinates that may be written down from those of bf by cyclic change of x, y, z and a, b, c.

These six points lie on a conic Σ_3 with equation of the form (3.1) where

$u = 2b^4c^4, v = 2c^4a^4, w = 2a^4b^4, f = 11a^4b^2c^2, g = 11a^2b^4c^2, h = 11a^2b^2c^4.$ (7.1)

8. Intersection of sides of triangles ABC and DEF and the conic on which they lie

Point fa = AB^EF has co-ordinates $(2a^2, b^2, 0)$. Points db = BC^FD and ec = CA^DE have co-ordinates which may be written down from those of fa by cyclic change of x, y, z and a, b, c.

Point fb = AB^FD has co-ordinates $(a^2, 2b^2, 0)$. Points dc = BC^DE and ea = CA^EF have co-ordinates which may be written down from those of fb by cyclic change of x, y, z and a, b, c.

These six points lie on a conic Σ_4 with equation of the form (3.1) where
$u = 4b^4c^4$, $v = 4c^4a^4$, $w = 4a^4b^4$, $f = -5a^4b^2c^2$, $g = -5a^2b^4c^2$, $h = -5a^2b^2c^4$ (8.1)

9. The Brocard axis and the centres of conics Σ_1 - Σ_4

The Brocard axis OK has equation

$$b^2c^2(b^2 - c^2)x + c^2a^2(c^2 - a^2)y + a^2b^2(a^2 - b^2)z = 0\ldots \quad (9.1)$$

A conic with equation (3.1) is known to have a centre with x – co-ordinate

$$x = vw - gv - hw - f^2 + fg + hf \quad (9.2)$$

with y- and z- co-ordinates following from (9.2) by cyclic change of u, v, w and f, g, h.

It may now be shown that the conics Σ_1, Σ_2 , Σ_3, Σ_4 have x-co-ordinates

Σ_1: $a^2(3(b^2 + c^2) - 5a^2)$, (9.3)

Σ_2: $a^2(7(b^2 + c^2) - 11a^2)$, (9.4)

Σ_3: $a^2(11(b^2 + c^2 - 13a^2)$, (9.5)

Σ_4: $a^2(5(b^2 + c^2) - a^2)$. (9.6)

It may now be checked that these four points all lie on the Brocard axis, as of course do the centres of the 7-point circle and the Brocard ellipse (that touches all six sides of the two triangles.

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