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Harmonic ranges in a coaxal system of circles

Christopher J Bradley

1. Introduction

Suppose we are given a circle that cuts the sides BC, CA, AB of a triangle ABC at pairs of points on each side, say P and L on BC, Q and M on CA and R and N on AB, in each case in the specified order working around the triangle in an anticlockwise direction. Now let P' and L' be the harmonic conjugates of P and L with respect to B and C, Q' and M' the harmonic conjugates of Q and M with respect to C and A and R' and N' the harmonic conjugates of R and N with respect to A and B. First we prove that P', L', Q', M', R' and N' always lie on a conic. We then investigate the conditions on the given circle for this conic also to be a circle.

We define such a pair of circles as a *harmonic pair of circles* and it turns out that each such pair belongs to the coaxal system of circles of which the polar circle, centre the orthocentre H of ABC and the circumcircle, centre the circumcentre O of ABC may be regarded as defining members. Since the polar circle is real only when ABC is obtuse we remark that other circles in this coaxal system are the nine-point circle, centre T, the midpoint of OH, and the orthocentroidal circle based on GH as diameter, where G is the centroid of ABC. If a harmonic pair S and S' have centres at X and X', then X and X' divide O and H harmonically. See Fig. 1.

Investigation of the equations of the circles forming a harmonic pair enables us to relate their equations in a canonical form in terms of those of the polar circle and the circumcircle, with the position of their centres. Analysis that is first applied to the coaxal system of circles with centres on the Euler line may be transferred to the circles of any coaxal system.

If one sets up a more general figure in which conics replace circles and one draws the lines PN, QL, RM to form a triangle DEF and one draws the lines P'N', Q'L', R'M' to form a triangle UVW then we prove that triangles ABC, DEF, UVW are mutually in perspective, and moreover the three Desargues's lines of the three perspectives coincide and the three perspectors are collinear. This is possibly a known result, but as its proof is straightforward we include it thereby providing an extension of the results about harmonic pairs of circles.

Areal co-ordinates are used throughout the article.

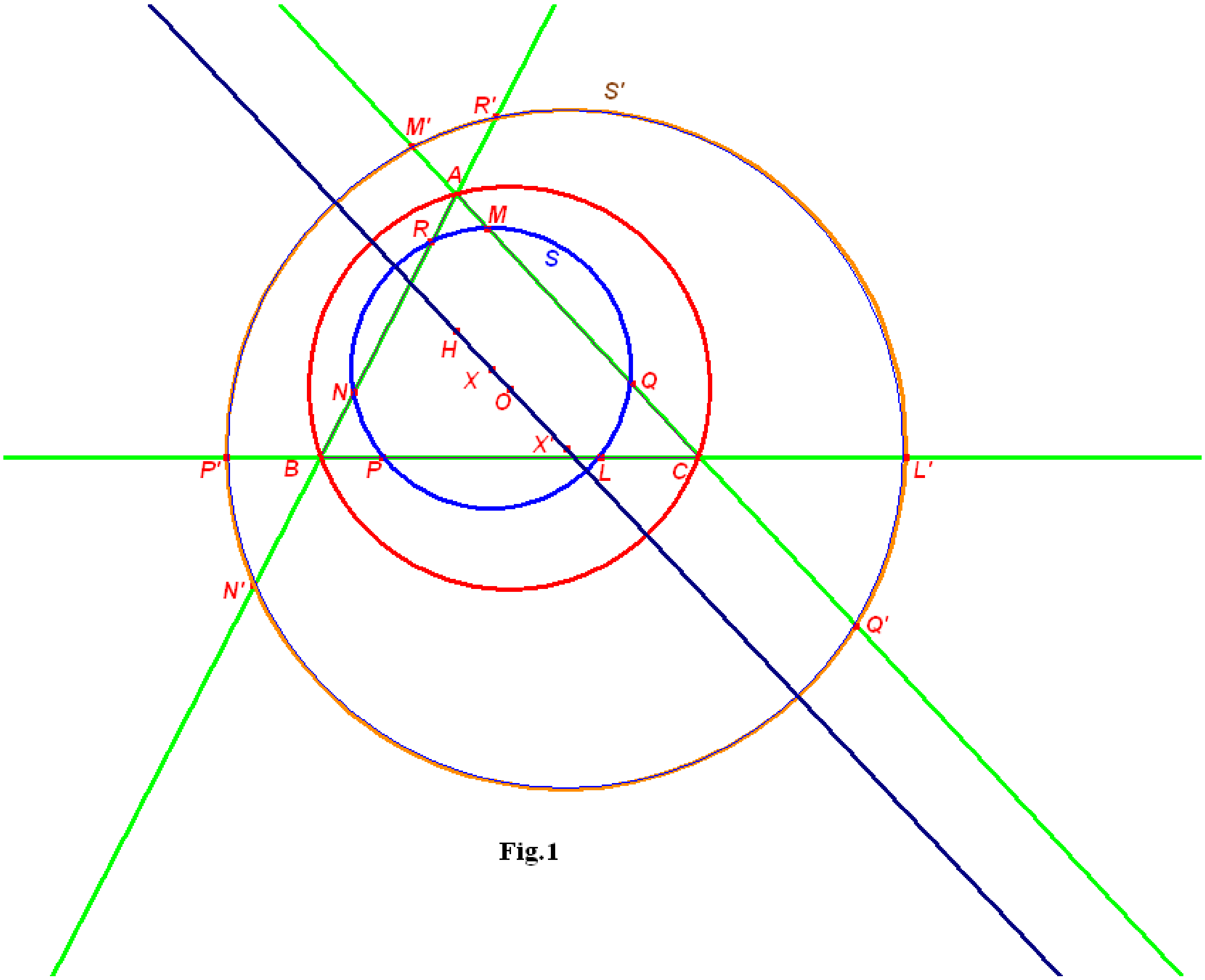

Fig.1

2. -Harmonic conjugate points lie on a conic

Suppose a conic cuts the sides BC, CA, AB of a triangle in points P, L on BC, Q, M on CA and R, N on AB, then by Carnot's theorem

$$(BP/PC)(BL/LC)(CQ/QA)(CM/MA)(AR/RB)(AN/NB) = 1. \tag{2.1}$$

If P' is the harmonic conjugate of P with respect to B and C, and L', Q', M', R', N' are similarly defined, then we have (BP'/P'C) = – (BP/PC) etc. And so from Equation 1.1,

$$(BP'/P'C)(BL'/L'C)(CQ'/Q'A)(CM'/M'A)(AR'/R'B)(AN'/N'B) = 1. \tag{2.2}$$

Now, if L, M, N and P, Q, R are the feet of two Cevians, then L'M'N' and P'Q'R' are two straight lines, by the converse of Menelaus's theorem, and hence form a degenerate conic. Otherwise the conditions for the converse of Carnot's theorem are satisfied and P', L', Q', M', R', N' lie on a non-degenerate conic.

The degenerate case occurs in at least one familiar case when the conic is the nine-point circle, and the harmonic conjugates then lie on two straight lines, one being the line at infinity.
This is perhaps a good place to mention that if we work in the complex field the theorems in this article are true irrespective of whether the points of intersection of the conic cut the triangle sides in real points or not.

3. The condition for the harmonic conjugate points to lie on a circle

Suppose that P and L have co-ordinates (0, q, r) and (0, m, n) respectively. Then if we put $x = 0$ in the equation of the given conic it must reduce to $(ry - qz)(ny - mz) = 0$. Since P' and L' are the harmonic conjugates of P and L, their co-ordinates must be (0, q, – r) and (0, m, – n) and it follows that if we put $x = 0$ in the equation of the harmonically related conic it must reduce to $(ry + qz)(ny + mz) = 0$. Thus under harmonic conjugation $rny^2 - (rm + qn)yz + qmz^2$ becomes $rny^2 + (rm + qn)yz + qmz^2$ and the change is effected simply by altering the sign of the yz term.

If then the original conic has equation

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad (3.1)$$

it follows immediately that the harmonically related conic has equation

$$ux^2 + vy^2 + wz^2 - 2fyz - 2gzx - 2hxy = 0. \qquad (3.2)$$

We now investigate the conditions under which both these equations represent circles. From Section 2.3.10 of Bradley [1] there must exist a constant k such that the following six equations hold.

$$v + w - 2f = a^2,\ \ w + u - 2g = b^2,\ \ u + v - 2h = c^2$$
$$v + w + 2f = -ka^2,\ w + u + 2g = -kb^2,\ u + v + 2h = -kc^2. \qquad (3.3)$$

The solutions of Equation (2.3) are

$$f = -\tfrac{1}{4}a^2(k + 1),\ g = -\tfrac{1}{4}b^2(k + 1),\ h = -\tfrac{1}{4}c^2(k + 1),$$
$$u = \tfrac{1}{4}(b^2 + c^2 - a^2)(-k + 1),\ v = \tfrac{1}{4}(c^2 + a^2 - b^2)(-k + 1),\ w = \tfrac{1}{4}(a^2 + b^2 - c^2)(-k + 1). \qquad (3.4)$$

From Section 2.3.6 of Bradley [1] the co-ordinates of the centre of the conic with Equation (3.1) are proportional to

$$(vw - gv - hw - f^2 + fg + fh,\ wu - hw - fu - g^2 + gh + gf,\ uv - fu - gv - h^2 + hf + hg).$$

Substituting values from Equation (3.4) we find the x-co-ordinate to be proportional to

$$-2a^4k + a^2(1 + k)(b^2 + c^2) - (-k + 1)(b^2 - c^2)^2 \qquad (3.5)$$

with y- and z-co-ordinates found from (3.5) by cyclic change of a, b, c.

Now the equation of the Euler line of triangle ABC is

$$(b^2 - c^2)(b^2 + c^2 - a^2)x + (c^2 - a^2)(c^2 + a^2 - b^2)y + (a^2 - b^2)(a^2 + b^2 - c^2)z = 0. \qquad (3.6)$$

Substituting the co-ordinates of the centre of the circle into Equation (2.6) we find the centre lies on this line. Similar analysis holds for the conic with Equation (3.2). It follows that whatever the value of k the centres of the two circles must lie on the Euler line.

4. **The coaxal system formed by harmonic pairs of circles**

From Section 3 we know that the equations of a harmonic pair of circles are of the form

$$(-k + 1)\{(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2\} - 2(k + 1)\{a^2yz + b^2zx + c^2xy\} = 0, \qquad (4.1)$$

and

$$(-k + 1)\{(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2\} + 2(k + 1)\{a^2yz + b^2zx + c^2xy\} = 0, \qquad (4.2)$$

Note that one may obtain Equation (4.2) from Equation (4.1) by replacing k by 1/k.

First let us look at some particular cases. When k = – 1 one gets the polar circle with equation

$$(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2 = 0 \qquad (4.3)$$

from both Equations (4.3) and (4.4). It follows that the polar circle is self-conjugate being its own harmonic partner. Note that the polar circle is real if and only if triangle ABC is obtuse, so if one wishes to visualise cases for all real k one must draw ABC as obtuse. Similarly if one puts k = 1 one gets the circumcircle with equation

$$2(a^2yz + b^2zx + c^2xy) = 0, \qquad (4.4)$$

again from both Equations (4.1) and (4.2). (The factor 2 is introduced in Equation (4.4) rather than in Equations (4.7) and (4.8) below.) It follows that the circumcircle is also self-conjugate. If one puts k = 0, Equation (4.1) becomes the nine-point circle with equation

$$\{(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2)z^2\} - 2\{a^2yz + b^2zx + c^2xy\} = 0. \qquad (4.5)$$

Equation (4.2) factorizes and reduces to the line pair

$$x + y + z = 0 \text{ and } (b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z = 0. \qquad (4.6)$$

The first of these is the line at infinity and the second line we shall identify very soon.

Abbreviating equations (4.3) and (4.4) to read $S_P = 0$ and $S_C = 0$ we see that Equations (4.1) and (4.2) may be written in the form

$$(1 - k)S_P - (1 + k)S_C = 0 \tag{4.7}$$

and

$$(1 - 1/k)S_P - (1 + 1/k)S_C = 0 \tag{4.8}$$

respectively. It follows that a pair of circles are harmonic if, and only if, their centres are on the Euler line and they are in the coaxal system of circles defined by the polar circle and the circumcircle. For an obtuse-angled triangle ABC, these two circles define an intersecting system of coaxal circles and the second of Equations (4.6) is therefore their common chord. In the case of an acute-angled triangle, when working over the real field, one may view the coaxal system as a non-intersecting system of circles, as in Fig.1, defined by the circumcircle and the line corresponding to the second of Equations (4.6), which is now the radical axis.

5. The location of the centres of a harmonic range of circles

Circle centres may be computed using the formula from Section 2.3.6 of Bradley [1]. Details of the calculation, carried out by *DERIVE*, are omitted. Only the x-co-ordinate of each point needs to be recorded, since the y- and z-co-ordinates follow by cyclic change of a, b, c. Co-ordinates are normalized, as is necessary for deductions to be made about the disposition of the centres.

For the polar circle the centre is at H with x-co-ordinate

$$x_P = (a^2 + b^2 - c^2)(a^2 + c^2 - b^2)/(a + b + c)(b + c - a)(c + a - b)(a + b - c). \tag{5.1}$$

For the circumcircle the centre is at O with x-co-ordinate

$$x_C = a^2(b^2 + c^2 - a^2)/(a + b + c)(b + c - a)(c + a - b)(a + b - c). \tag{5.2}$$

For the circle with Equation (4.1) the x-co-ordinate is

$$x_k = (1/2)(1 - k)x_P + (1/2)(1 + k)x_C. \tag{19}$$

For the circle with Equation (4.2) the x-co-ordinate is

$$x_{(1/k)} = (1/2)(1 - 1/k)x_P + (1/2)(1 + 1/k)x_C. \tag{20}$$

Since $\{x_P, x_C; x_k, x_{(1/k)}\} = \{-1, 1; k, 1/k\} = -1$ the centres of any harmonic pair of circles separate the centre of the polar circle and the centre of the circumcircle harmonically.

Another circle in this coaxal system is the orthocentroidal circle, which is the circle with GH as diameter. It has parameter $k = -1/3$. Circles centre G, the centroid, and deL, deLongchamps point, form a harmonic pair with $k = 1/3$ and $k = 3$ respectively.

6. Perspectives formed by harmonic pairs of circles

The results of this section do not depend on the conics involved being circles, so we establish the results in the more general case. The notation used, however, is the same as before, with a conic meeting sides BC, CA, AB respectively at pairs of points P, L; Q, M, R, N. The harmonic conjugate points are denoted by P', L', Q', M', R', N' and as we have seen in Section 2 these points also lie on a conic. Suppose now that lines PN, QL, RM are drawn to form triangle DEF with D = PN^QL etc. and lines P'N', Q'L', R'M' are drawn to form triangle UVW with U = P'N'^Q'L' etc.

Result 1 Triangles ABC and DEF are in perspective

Proof
BC^EF is the same as RM^PL = I, say. CA^FD is the same as QM^PN = J, say. AB^DE is the same as RN^QL = K, say. It follows, by Pascal's theorem for the hexagon (RMQLPN) inscribed in the conic S that IJK is a straight line, which is therefore the Desargues's axis of perspective for triangles ABC and DEF. Using the notation of Fig. 2 the vertex of perspective is the point Z.

Result 2 Triangles ABC and UVW are in perspective

Proof
Since P'L' is the same line as PL, Q'M' is the same line as QM and R'N' is the same line as RN the proof of Result 2 follows by the same reasoning as for Result 1, but with the hexagon (R'M'Q'L'P'N') inscribed in the conic S'. Furthermore the intersections are again the points I, J, K which is the same Desargues's axis as before. The vertex of perspective is shown as the point Y in Fig. 2.

Result 3 Triangles DEF and UVW are in perspective

Proof
Since VW^EF = RM^R'M' = I etc., Result 3 is immediate and once again the Desargues' axis is the same line. The vertex of perspective is shown as X in Fig. 2.

Result 4 The perspectors X, Y, Z are collinear

Proof
By using a projective transformation we may map the line IJK to the line at infinity and then the three triangles map into three triangles homothetic in pairs. The question now reduces to whether the three homothety centres are collinear. Denote the three pairwise homothetic triangles by T_1,

T_2, T_3 and let the homothety θ_{ij} carrying T_i to T_j have centre C_{ij}. Now θ_{12} followed by θ_{23} leaves the line joining $C_{12}C_{23}$ invariant. This composition is θ_{13} so C_{13} is on this line.

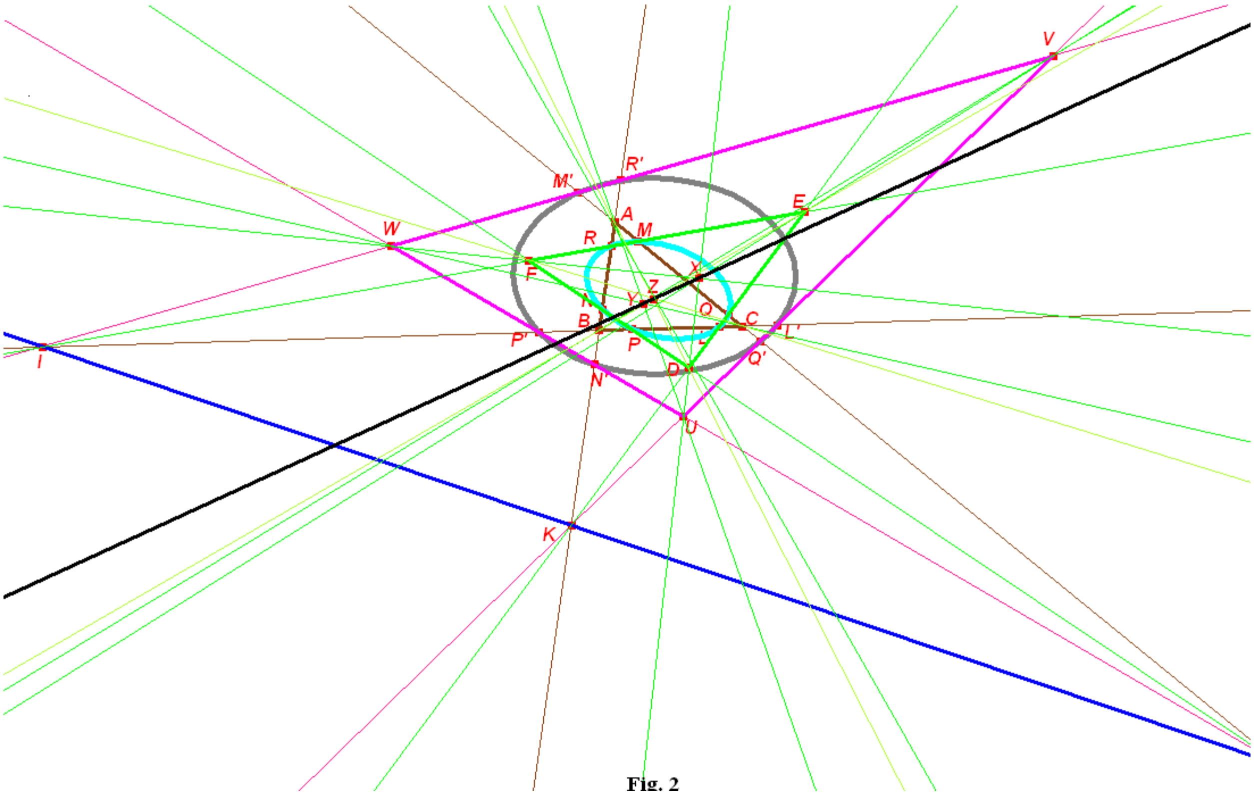

Fig. 2

I am grateful to Dr Geoff. Smith of the University of Bath for providing the proof of Result 4 and for having a number of useful conversations with me concerning the contents of this article.

Reference

1. C.J.Bradley, *The Algebra of Geometry*, Highperception, Bath 2007.

Flat 4,
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP.