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Two In-Perspective Triangles inscribed in a Conic

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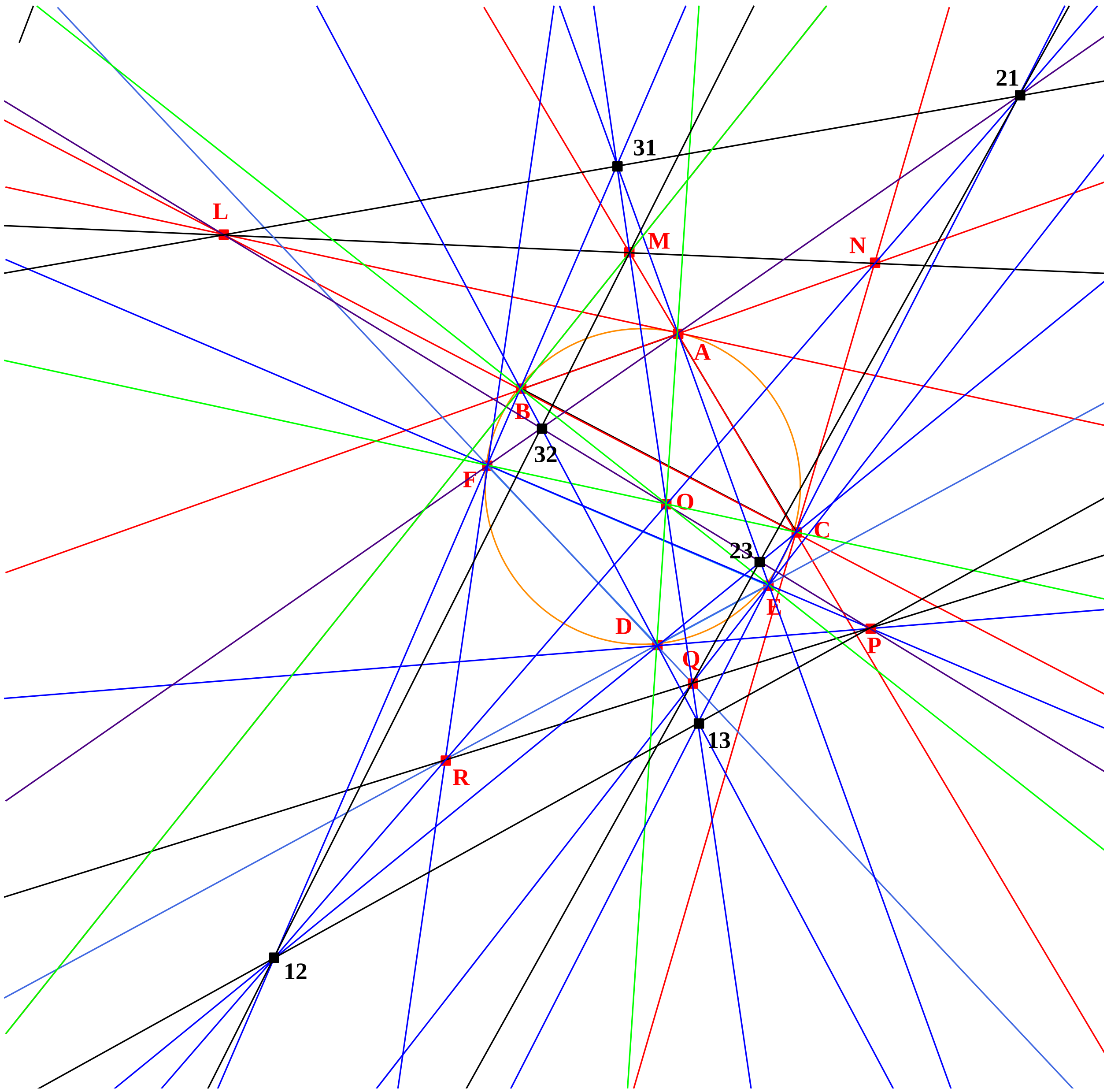

Fig. 1

In-perspective Triangles inscribed in a Conic and Points and Lines of significance

Abstract: Triangles ABC and DEF are inscribed in a conic and are in perspective through a point O. Polar lines LMN and PQR are drawn and six points of concurrency are shown to define a number of important collinearities. Two conics also emerge.

1. Setting the Scene

We use areal co-ordinates with ABC as triangle of reference. We take the circumconic to have equation

$$fyz + gzx + hxy = 0 \qquad (1.1)$$

and the point O to have co-ordinates O(u, v, w).

The equation of AO is

$$wy = vz \qquad (1.2)$$

Line AO meets the circumconic again at D with co-ordinates (x, y, z), where

$$x = fvw,\ y = -v(gw + hv),\ z = -w(gw + hv). \qquad (1.3)$$

Similarly E has co-ordinates

$$x = -u(hu + fw),\ y = gwu,\ z = -w(hu + fw). \qquad (1.4)$$

and F has co-ordinates

$$x = -u(fv + gu),\ y = -v(fv + gu),\ z = huv. \qquad (1.5)$$

The equations of the tangents at the vertices are:

A:	$hy + gz = 0.$	(1.6)
B:	$fz + hx = 0.$	(1.7)
C:	$gx + fy = 0.$	(1.8)
D:	$(gw + hv)^2x + fgw^2y + hfv^2z = 0.$	(1.9)
E:	$fgw^2x + (hu + fw)^2y + ghu^2z = 0.$	(1.10)
F:	$hfv^2x + ghu^2y + (gu + fv)^2z = 0.$	(1.11)

The equation of EF is

$$fvwx - u(fw + hu)y - u(fv + gu)z = 0. \qquad (1.12)$$

The equations of FD and DE may be obtained from equation (1.12) by cyclic change of x, y, z and f, g, h and u, v, w.

The tangent at A meets BC at L (0, – g, h). The tangent at B meets CA at M (f, 0, – h) and the tangent at C meets AB at N (– f, g, 0).

The tangent at D meets EF at P with co-ordinates (x, y, z), where

$$x = fu(gw - hv),\ y = -hfv^2 - gu(gw + hv),\ z = fgw^2 + hu(gw + hv) \qquad (1.13)$$

The co-ordinates of Q and R follow from equations (1.13) by cyclic change of x, y, z and f, g, h and u, v, w.

2. The polar lines

The equation of LMN is

$$x/f + y/g + z/h = 0. \tag{2.1}$$

The equation of PQR is

$$f(g^2w^2 + ghvw + h^2v^2)x + g(h^2u^2 + hfwu + f^2w^2)y + h(f^2v^2 + fguv + g^2u^2) z = 0. \tag{2.2}$$

The equations of LP, MQ, NR are

$$(gw + hv)x = u(hy + gz), \tag{2.3}$$
$$(hu + fw)y = v(fz + hx), \tag{2.4}$$
$$(fv + gu)z = w(gx + fy). \tag{2.5}$$

It may now be checked that O lies on LP, MQ and NR.

3. Six lines and six more points

The equations of the lines joining vertices of the two triangles are:

AE: $(fw + hu)y + guz = 0,$ (3.1)
BF: $(gu + fv)z + hvx = 0,$ (3.2)
CD: $(hv + gw)x + fwy = 0,$ (3.3)
AF: $huy + (fv + gu)z = 0,$ (3.4)
BD: $fvz + (gw + hv)x = 0,$ (3.5)
CE: $gwx + (hu + fw)y = 0.$ (3.6)

Points of intersection of these lines have labels and co-ordinates as follows:

CD^AE = 23 $x = fgwu,\ y = – gu(gw + hv),\ z = (fw + hu)(gw + hv),$ (3.7)
AE^BF = 31 $x = (fv + gu)(fw + hu),\ y = ghuv,\ z = – hv(fw + hu),$ (3.8)
BF^CD = 12 $x = – fw(fv + gu),\ y = (fv + gu)(gw + hv),\ z = hfvw,$ (3.9)
CE^AF = 21 $x = (fv + gu)(fw + hu),\ y = – gw(fv + gu),\ z = ghwu,$ (3.10)
AF^BD = 32 $x = hfuv,\ y = (gw + hv)(gu + fv),\ z = – hu(gw + hv),$ (3.11)
BD^CE = 13 $x = – fv(hu + fw),\ y = fgvw,\ z = (hu + fw)(hv + gw).$ (3.12)

4. Additional points of concurrency and collinearity

It may now be shown that 32 and 23 both lie on LP and that 13 and 31both lie on MQ and that 21 and 12 both lie on NR.

The line 31 21 has equation

$$fghvwx + (fv + gu)(fw + hu)(hy + gz) = 0. \qquad (4.1)$$

It may now be shown that L lies on the line 31 21. Similarly M lies on line 12 32 and N lies on line 23 13.

The equation of line 12 13 is

$$f((g^2w^2 + ghuv + h^2v^2) + ghu(gw + hv))x + f(gw(fw + hu)y + hv(fv + gu))z = 0. \qquad (4.2)$$

It may now be shown that P lies on the line 12 13. Similarly Q lies on line 23 21 and R lies on line 31 32.

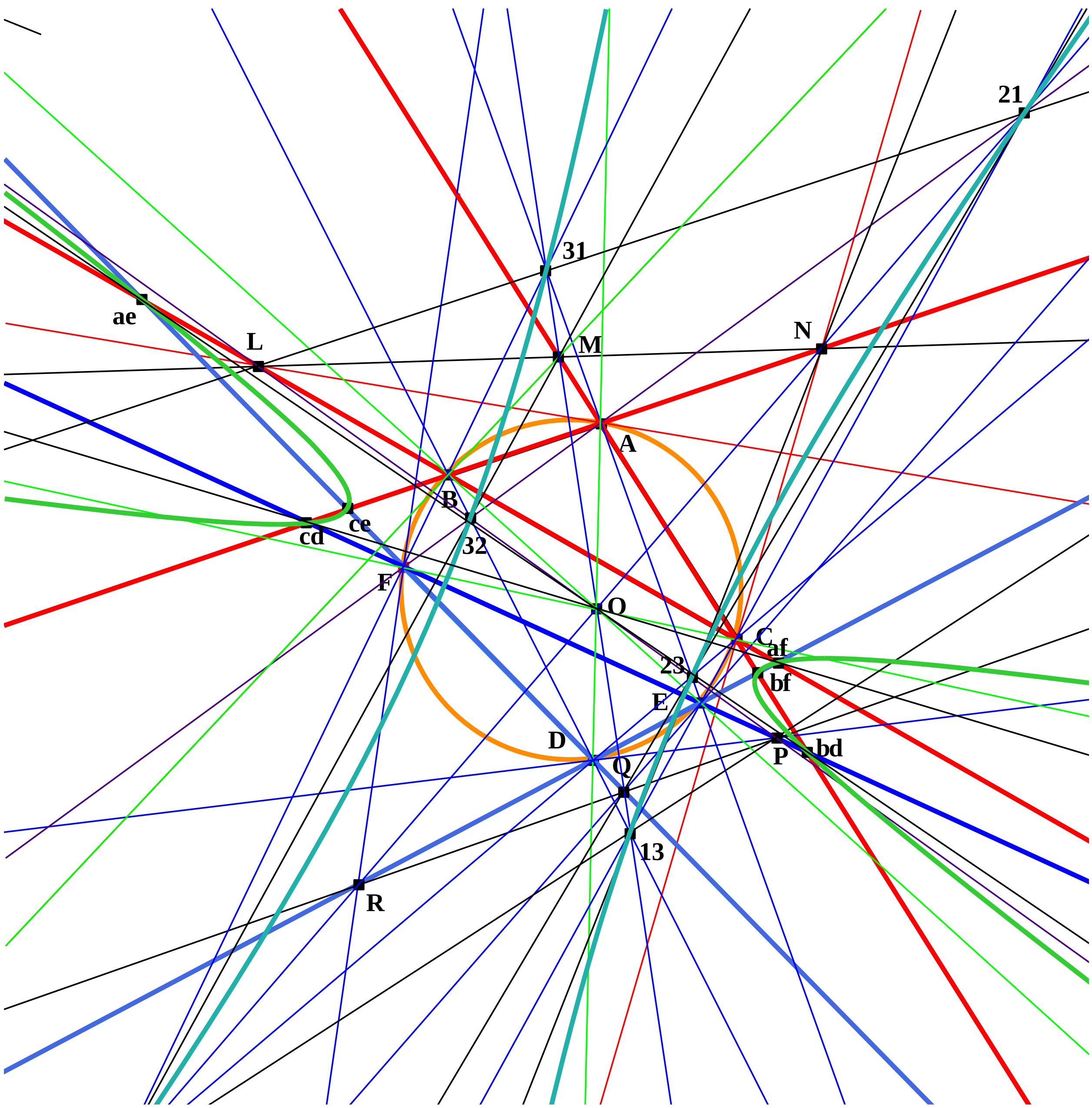

Fig.2

5. The two conics

In Fig.2 we show, in addition to the material from Fig.1, two conics, one expected and the other new as far as we are concerned. The first and unexpected conic passes through the six points we have just been describing, that is 12 21 23 32 31 13.

It has the form

$$ax^2 + by^2 + cz^2 + 2pyz + 2qzx + 2rxy = 0, \quad (5.1)$$

where we find

$$a = 2fghvw(gw + hv), \quad (5.2)$$
$$b = 2fghwu(hu + fw), \quad (5.3)$$
$$c = 2fghuv(fv + gu), \quad (5.4)$$
$$p = fk,\ q = gk,\ r = hk, \quad (5.5)$$

where

$$k = f^2gvw^2 + f^2hv^2w + fg^2u\,w^2 + fh^2uv^2 + g^2hu^2w + gh^2u^2v + fghuvw. \quad (5.6)$$

We now define the six points where triangles ABC and DEF intersect. They are

ae = BC^FD: $x = 0$, $y = v(fv+ gu)$, $z = gwu$, (5.7)
cd = AB^EF : $x = u(fw + hu)$, $y = fvw$, $z = 0$, (5.8)
bf = CA^DE: $x = huv$, $y = 0$, $z = w(gw + hv)$, (5.9)
af = BC^DE: $x = 0$, $y = huv$, $z = w(fw + hu)$, (5.10)
bd = CA^EF: $x = u(gu + fv)$, $y = 0$, $z = fvw$, (5.11)
ce = AB^FD: $x = gwu$, $y = v(gw + hv)$, $z = 0$. (5.12)

These points, as expected lie on a conic of the form (5.1), with

$$a = 2fv^2w^2(gw + hv), \quad (5.13)$$
$$b = 2gw^2u^2(fw + hu), \quad (5.14)$$
$$c = 2hu^2v^2(fv + gu), \quad (5.15)$$
$$p = – uvw(f^2vw + fu(gw + hv) + 2ghu^2), \quad (5.16)$$
$$q = – uvw (g^2wu + gv(hu + fw) + 2hfv^2), \quad (5.17)$$
$$r = – uvw(h^2uv + hw(fv + gu) + 2fgw^2). \quad (5.18)$$

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