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Basic Properties of a Quadrangle possessing an Incircle

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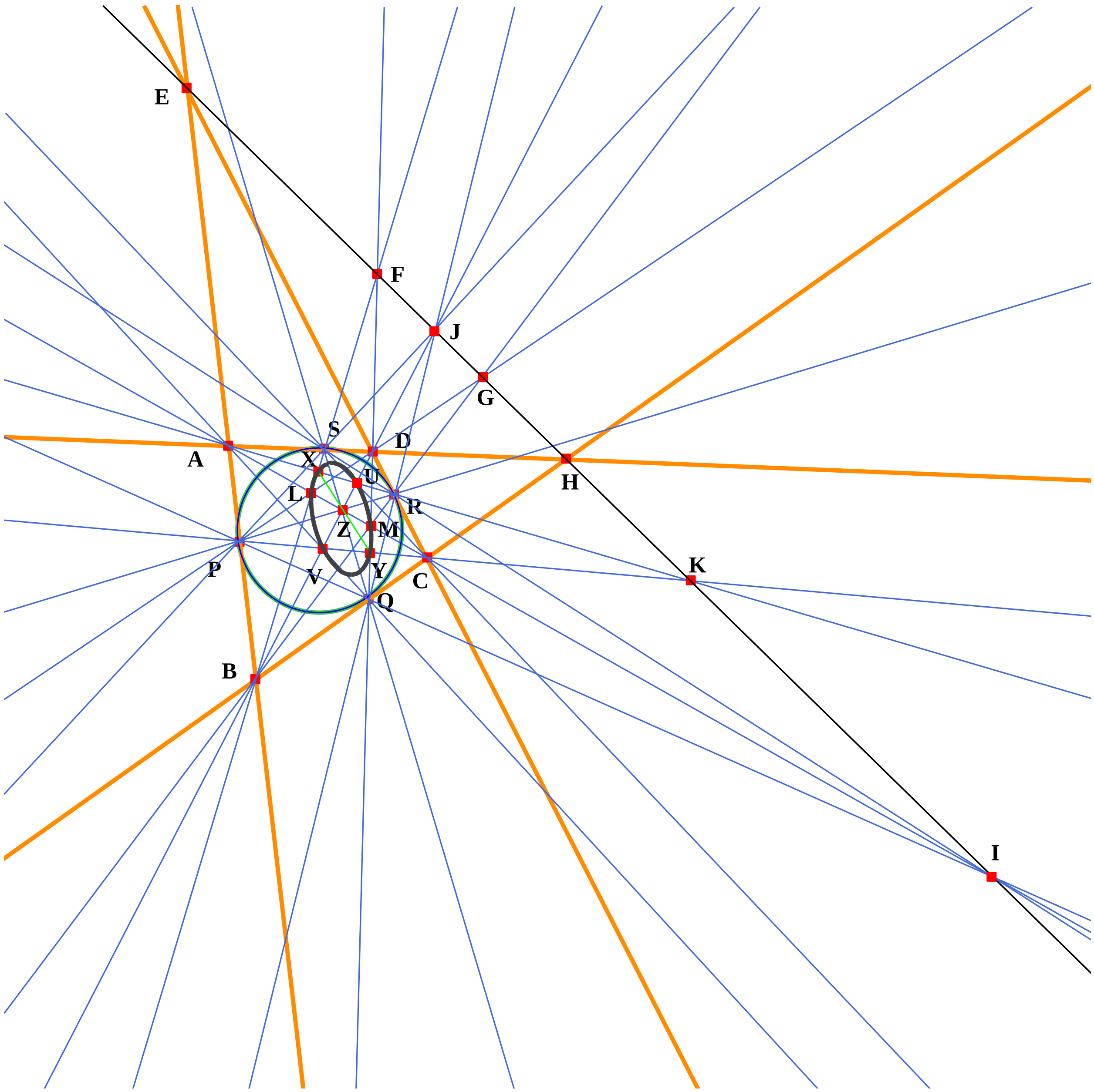

Fig. 1

Quadrangle and Incircle with associated Conic and their Polar Line

1. Introduction

Starting with a circle PQRS tangents are drawn at P, Q, R, S to produce a quadrangle ABCD with P on AB etc. The intersection of the diagonals AC and BD is the point labelled Z. Tangents

AB and CD intersect at E and tangents AD and BC meet at H. By construction it follows that EH is the polar of Z with respect to circle PQRS. Point X = AR^BS, point Y = CP^DQ. We prove that XYZ is a straight line and that Z lies on both PR and QS. We prove that lines DP, AC, BS are concurrent at a point L and that lines BR, AC, DQ are concurrent at a point M. We also show that lines AR, BD, CS are concurrent at a point U and that lines AQ, BD, CP are concurrent at V.

It then follows that points X, Y, L, M, U, V lie on a conic, showing that every quadrangle with an incircle also possesses an interior conic Σ. It also turns out that EH is also the polar of Z with respect to Σ. Finally we show that there are five other significant points on the polar line. These are F = BS^DQ, J = PS^QR, G = DP^BR, K = AR^CP, I = SR^PQ. It is also shown that J lies on BD and I lies on AC.

Proofs are given using areal co-ordinates with PQR as triangle of reference.

2. Points P, Q, R, S, A, B, C, D, Z

Points P, Q, R form the triangle of reference, so we take S to be a general point on the circumcircle with equation

$$a^2yz + b^2zx + c^2xy = 0, \qquad (2.1)$$

and therefore having co-ordinates $S(-a^2t(1-t), b^2(1-t), c^2t)$. In Fig. 1 we have chosen $1 < t < \infty$, so that PQRS appear in that cyclic order.

The equation of the tangent at P which is the line AB has equation

$$c^2y + b^2z = 0. \qquad (2.2)$$

The equation of the tangent at Q which is the line BC has equation

$$a^2z + c^2x = 0. \qquad (2.3)$$

The equation of the tangent at R which is the line CD has equation

$$b^2x + a^2y = 0. \qquad (2.4)$$

The equation of the tangent at S which is the line DA has equation

$$x/a^2 + t^2y/b^2 + (1-t)^2z/c^2 = 0. \qquad (2.5)$$

The co-ordinates of the vertices of the quadrangle are therefore:

A: $x = a^2(2t-1), y = -b^2, z = c^2$. (2.6)

B: $x = a^2, y = b^2, z = -c^2$. (2.7)

C: $x = -a^2, y = b^2, z = c^2$. (2.8)

D: $x = -a^2(1-t), y = b^2(1-t), z = c^2(1+t)$. (2.9)

The equation of AC is

$$x/a^2 + ty/b^2 + (1 – t)z/c^2 = 0, \qquad (2.10)$$

and the equation of BD is

$$x/a^2 – ty/b^2 + (1 – t)z/c^2 = 0. \qquad (2.11)$$

The point Z = AC^BD therefore has co-ordinates $Z(– a^2(1 – t), 0, c^2)$ (2.12)

Line PR has equation y = 0 and line QS has equation

$$c^2x + a^2(1 – t)z = 0. \qquad (2.13)$$

It follows immediately that Z lies on both PR and QZ.

3. Points U, V, L, M, X, Y and the interior conic Σ passing through these points

The equation of AQ is

$$c^2x + a^2(1 – 2t)z = 0. \qquad (3.1)$$

The point V is the intersection of this line with BD (equation (2.11)) and so has co-ordinates

$$x = a^2(2t – 1),\ y = b^2,\ z = c^2. \qquad (3.2)$$

The equation of the line CP is

$$b^2z = c^2y, \qquad (3.3)$$

from which we observe that V lies on CP also.

The line AR has equation

$$b^2x = a^2(1 – 2t)y. \qquad (3.4)$$

The point U is the intersection of this line with BD (equation (2.11)) and so has co-ordinates

$$x = a^2(1 – t)(1 – 2t),\ y = b^2(1 – t),\ z = c^2(3t – 1). \qquad (3.5)$$

The line CS has equation

$$(1 – 2t)x/a^2 – t^2y/b^2 + (1 – t)^2z/c^2 = 0. \qquad (3.6)$$

It may now be checked that U also lies on CS.

The equation of BS is

$$x/a^2 = t^2y/b^2 + (t^2 – 1)z/c^2. \qquad (3.7)$$

The point L is the intersection of this line with AC (equation (2.10)) and so has co-ordinates

$$x = a^2(2t^2 – t – 1),\ y = b^2(1 – t),\ z = c^2(1 + t) \qquad (3.8)$$

The equation of the line DP is

$$b^2(1 – t)z = c^2(1 + t)\ y. \qquad (3.9)$$

It is now obvious that L also lies on DP.

The equation of the line BR is

$$b^2x = a^2y. \qquad (3.10)$$

The point M is the intersection of this line with AC (equation (2.10)) and so has co-ordinates

$$x = a^2(t – 1),\ y = b^2(t – 1),\ z = c^2(t + 1). \qquad (3.11)$$

The equation of the line DQ is

$$c^2(1 + t)x + a^2(1 – t)z = 0. \qquad (3.12)$$

It is now obvious that M lies also on DQ.

The lines BS and AR (equations (3.7) and (3.4)) meet at the point X, which therefore has co-ordinates

$$x = a^2(t^2 – 1)(2t – 1),\ y = b^2(1 – t^2),\ z = c^2(t^2 + 2t – 1). \qquad (3.13)$$

The lines DQ (equation (3.12)) and CP (equation (3.3)) meet at the point Y, which therefore has co-ordinates

$$x = a^2(t – 1),\ y = b^2(t + 1),\ z = c^2(t + 1). \qquad (3.14)$$

It may now be checked that XYZ is a straight line.

The equation of the conic through U, V, L, M, X, Y is of the form

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad (3.15)$$

and after some computation we find

$$u = 2b^4c^4\ (1 + t),\ v = 2c^4a^4t\ (1 – t)^2,\ w = 2a^4b^4\ (1 + t)\ (1 – t)^2,$$
$$f = a^4b^2c^2t1 – t)\ (2t – 3),\ g = a^2b^4c^2\ (1 – t)\ (3t + 2),\ h = a^2b^2c^4t\ (t – 3). \qquad (3.16)$$

4. Points on the polar line

The tangents at P and R meet at the point E with co-ordinates E (a^2, – b^2, c^2). The tangents at Q and S meet at the point H with co-ordinates H (– a^2t, b^2 (2 – t), c^2t). The line EH has equation

$$x/a^2 + ty/b^2 + (t-1)z/c^2 = 0. \quad (4.1)$$

It is obvious that by construction that this line is the polar line of Z with respect to circle PQRS. It may also be checked that it is the polar line of Z with respect to the interior conic Σ. Other points lying on the polar line are F, G, I, J, K defined as follows:

F = BS^QD with co-ordinates F($a^2t(t-1)$, – $b^2(t^2 + t - t)$, $c^2t(1 + t)$). (4.2)
G = DP^BR with co-ordinates ($a^2(1-t)$, $b^2(1-t)$, $c^2(1 + t)$). (4.3)
I = PQ^RS with co-ordinates (a^2t, – b^2, 0). (4.4)
J = PS^QR with co-ordinates (0, $b^2(1-t)$, c^2t). (4.5)
K = AR^CP with co-ordinates ($a^2(1-2t)$, b^2, c^2). (4.6)

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