Article: CJB/2011/166

Analytic treatment of a Romanian problem

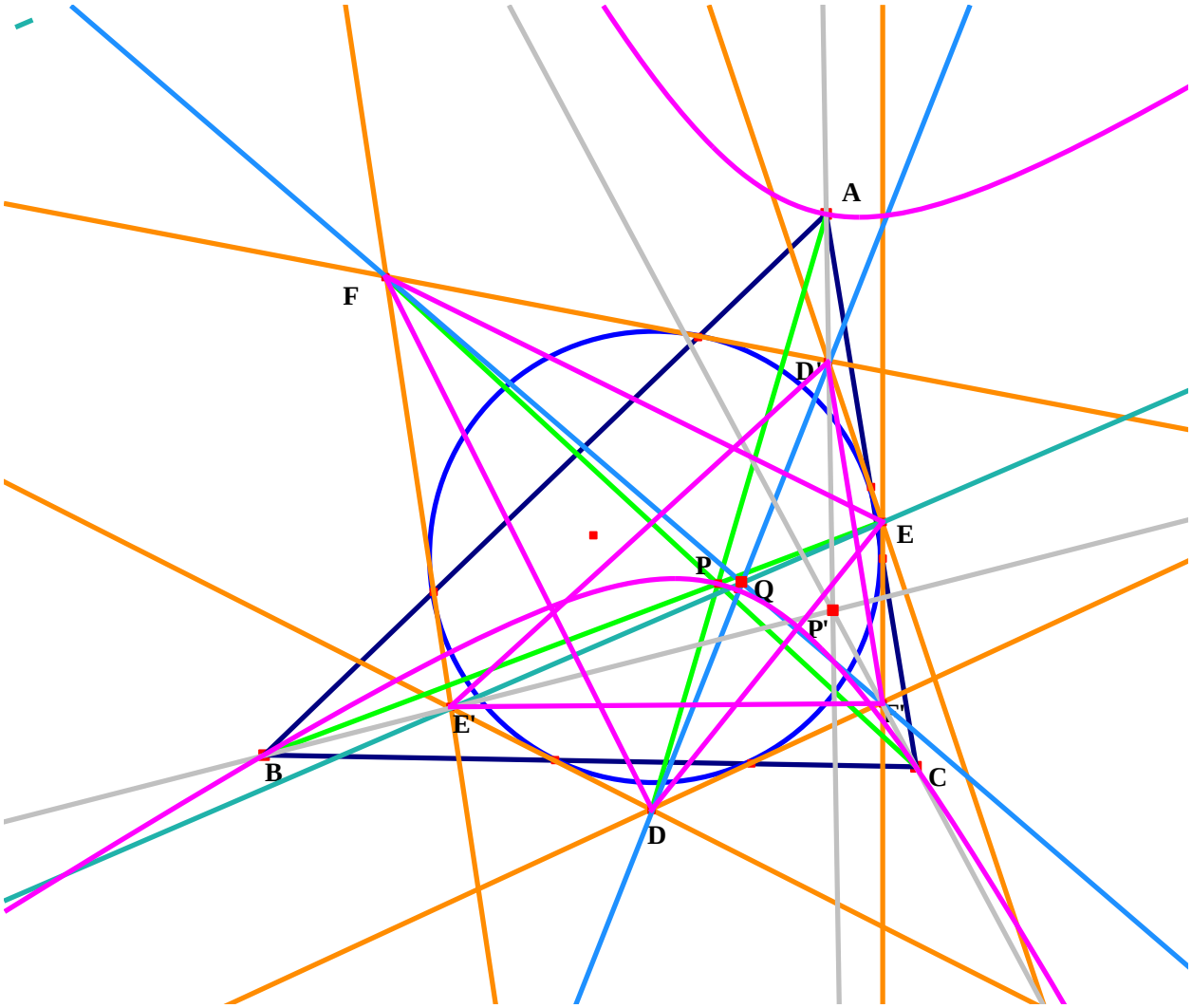

Fig. 1

Abstract: A triangle ABC is given and a conic that cuts each of its sides in two real points. The tangents at these points are drawn providing a hexagon of tangents circumscribing the conic, whose vertices are labelled DF'ED'FE as in Fig.1. It is shown that AD, BE, CF are concurrent as are AD', BE', CF' and DD', EE', FF'. The positions of the resulting perspectives P, P' and Q are located. In the Romanian question the conic was the nine-point circle and then P lies on the hyperbola $ABCKO_i$, where K is the symmedian point and O_i is the isotomic conjugate of the circumcentre O. The isotomic conjugate of P then lies on the diameter of the circumcircle joining the Steiner point and the Tarry point. It is also shown that in the general case AD', EF', FE' are concurrent as are BE', DF', FD' and CF', DE', ED'. Areal co-ordinates are used throughout with ABC the triangle of reference.

1. The conic and the points D, E, F and P

The abstract may be regarded as an introduction as it describes the situation completely. We suppose (for simplicity but therefore lacking in generality) that the conic passes through the feet of two Cevian points with co-ordinates (l, m, n) and (p, q, r). For the Romanian problem these are the midpoints and the feet of the altitudes

The equation of a conic Σ in areals is of the form

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \tag{1.1}$$

where the ratios of u, v, w, f, g, h may be determined by five points of the conic. We take the five points to be (0, m, n), (l, 0, n), (l, m, 0), (0, p, q) and (p, 0, r) with (p, q, 0) as a check. We find

$$u = 2mnqr,\ v = 2nlrp,\ w = 2lmpq,\ f = -\,lp(mr + nq),\ g = -\,mq(np + lr),\ h = -\,nr(lq + mp). \tag{1.2}$$

The tangent to Σ at the point (d, e, f) is

$$2mnqrdx + 2nlrpey + 2lmpqfz - nr\,(lq + mp)(ex + dy) - lp(mr + nq)(fy + ez) - mq(np + lr)(dz + fx) = 0. \tag{1.3}$$

It follows that the tangents at the points (0, m, n) and (0, q, r) on BC have equations

$$lp(mr - nq)(ny - mz) - mn(2lqr + mrp + npq)x = 0 \tag{1.4}$$

and

$$lp(nq - mr)(ry - qz) - qr(lmr + nlq + 2mnp)x = 0. \tag{1.5}$$

These tangents meet at D with co-ordinates (x, y, z), where

$$\begin{aligned} x &= -\,lp(mr - nq)^2, \\ y &= mq(lr(mr + 3nq) + np(3mr + nq)), \\ z &= nr(lq(3mr + nq) + mp(mr + 3nq)), \end{aligned} \tag{1.6}$$

The co-ordinates of E, F may be found from Equations (1.6) by cyclic change of x, y, z and l, m, n and p, q, r.

The equations of AD, BE, CF may now be obtained and it may then be shown that they are concurrent at a point P with co-ordinates (x, y, z), where

$$x = lp/\{l^2qr + 3lp(mr + nq) + mnp^2\}, \tag{1.7}$$

$$y = mq/\{m^2rp + 3mq(np + lr) + nlq^2\}, \tag{1.8}$$

$$z = nr/\{n^2pq + 3nr(lq + mp) + lmr^2\}. \tag{1.9}$$

A pure method of solution follows from Chasles' theorem that a triangle and its polar reciprocal are in perspective. Coolidge, J. L. *A Treatise on Algebraic Plane Curves.* New York: Dover, p. 33, 1959.

2. The points D', E', F' and P'

Consider the tangents that meet at D'. The one from a point on AB has equation

$$nqr(mp - lq)x + nrp(lq - mp)y - pq(2lmr + nlq + mnp) = 0. \qquad (2.1)$$

The one from a point on CA has equation

$$mnq(lr - np)x - nl(lqr + 2mrp + npq)y + lmq(np - lr)z = 0. \qquad (2.2)$$

These lines meet at the point D' with co-ordinates

$$(lp(3mr + nq), mq(lr - np), nr(mp - lq)). \qquad (2.3)$$

Consider next the tangents that meet at E'. (We suppose the Cevians are drawn as in Fig.1) The one from the point on BC has equation

$$lp(mr - nq)(ny - mz) - mn(2lqr + mrp + npq)x = 0. \qquad (2.4)$$

The one from the point on AB is

$$mnr(lq - mp)x + nlr(mp - lq)y - lm(lqr + mrp + 2npq)z = 0. \qquad (2.5)$$

These lines meet at the point E' with co-ordinates

$$(lp(nq - mr), - m(2mrp + lqr + npq), nr(lq - mp)). \qquad (2.6)$$

Consider next the tangents that meet at F'. The one from the point on BC has equation

$$lp(nq - mr)(ry - qz) - qr(lmr + nlq + 2mnp)x = 0. \qquad (2.7)$$

The one from the point on CA has equation

$$mqr(np - lr)x - rp(lmr + mnp + 2nlq)y + mpq(lr - np)z = 0. \qquad (2.8)$$

The equations of AD', BE', CF' are now easily written down and it may then be checked that they all pass through the point P' with co-ordinates

$$(lp(nq - mr), mq(np - lr), nr(lq - mp)). \qquad (2.9)$$

The lack of cyclic symmetry is due to the fact that D', E', F' depend on the positions of (l, m, n) and (p, q, r). (This is not the case for the point P.)

3. The lines DD', EE', FF' meet at a point Q

Using the co-ordinates obtained in Sections 1 and 2 we find the equations of DD', EE', FF' are as follows:

DD': $nq(l^2qr - mnp^2)x - nlp(2lqr + mrp + npq)y - lpq(lmr + nlq + 2mnp)z = 0.$ (3.1)

EE': $mn(3lr + np)x - nl(lr - np)y - lm(lr + 3np)z = 0.$ (3.2)

FF': $qr(lq + 3mp)x - rp(3lq + mp)y - pq(mp - lq)z = 0.$ (3.3)

The fact that these three lines are concurrent is a result of Brianchon's theorem. Johnson, R. A. §387 in *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle.* Boston, MA: Houghton Mifflin, p. 237, 1929.

The point of concurrence Q is found to have co-ordinates (x, y, z), where

$$\begin{aligned} x &= -lp(l^2qr(3mr + nq) + lp(m^2r^2 + 8mnqr - n^2q^2) + mnp^2(3mr + nq)), \\ y &= -mq(l^3qr^2 + 3l^2pr(mr + 2nq) + lnp^2(6mr + nq) - mn^2p^3), \\ z &= nr(l^3q^2r - l^2pq(6mr + nq) - 3lmp^2(mr + 2nq) - m^2np^3). \end{aligned} \tag{3.4}$$

There are, of course many other Brianchon points, but this is the only one we consider.

4. The position of P in the Balkan problem

This is when Σ is the nine-point circle and then (l, m, n) = (1, 1, 1) and (p, q, r) =

$$(1/(b^2 + c^2 - a^2), 1/(c^2 + a^2 - b^2), 1/(a^2 + b^2 - c^2)).$$

The co-ordinates of P are now

$$(1/(a^2(b^2 + c^2 - a^2) + b^2c^2), 1/(b^2(c^2 + a^2 - b^2) + c^2a^2), 1/(c^2(a^2 + b^2 - c^2) + c^2a^2)). \tag{4.1}$$

Cabri II plus and these co-ordinates show that P lies on the hyperbola $ABCKO_i$, where, K is the symmedian point and O_i is the isotomic conjugate of the circumcentre, which has equation

$$fyz + gzx + hxy = 0, \tag{4.1}$$

where

$$f = a^2(b^2 - c^2)(b^4 + c^4 + a^2b^2 + a^2c^2), \tag{4.2}$$

and g, h follow from f by cyclic change of a, b, c. In the above it has been proved that the isotomic conjugate of P lies on the line through the circumcentre, the third Brocard point and the Steiner and Tarry points. Gallatly, W. "The Steiner and Tarry Points." §143 in *The Modern Geometry of the Triangle, 2nd ed.* London: Hodgson, p. 102, 1913.
Details are left to the reader.

5. AD', EF', FE' are concurrent as are BE', DF', FD' and CF', DE', ED'

Finally there are three other concurrencies as indicated above. We find that AD', EF', FE' are concurrent at a point with co-ordinates (lp(mr + 3nq), mq(np – lr), nr(lq – mp). The co-ordinates of the other points of concurrence may be found by cyclic change of x, y, z and l, m, n and p, q, r.

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