Article 16

The Direct Similarity of the Miquel Point Configuration

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1. Introduction

Given a triangle *ABC* and points *L*, *M*, *N*, other than the vertices, lying on *BC*, *CA*, *AB* respectively, then circles *AMN*, *BNL*, *CLM* share a common point *P* called the Miquel point. For a given triangle the position of *P* obviously depends on the points *L*, *M*, *N*.

In this article we show that the centres *X*, *Y*, *Z* of the three circles form a triangle *XYZ* that is directly similar to *ABC* and that the centre of the similarity is the point *P* itself, by which we mean that *XYZ* is obtained from *ABC* by a rotation about *P* followed by dilation with centre *P*.

We also give an analysis of the converse result. What this amounts to is that given a triangle *ABC* and a point *P* not on the sides, then it is always possible to find circles centres *X*, *Y*, *Z* to provide a configuration having *P* as a Miquel point, but the options are more limited than one might suppose. A direct similarity centre *P* is involved and any angle of rotation (other than a right angle) is possible, but then the scale factor of the dilation is fixed and depends on the angle of rotation. When the angle of rotation is 0° the scale factor is ½ and triangle *LMN* is the pedal triangle of *P*.

We also consider the problem of where *L*, *M*, *N* have to be placed in order that the circles *AMN*, *BNL*, *CLM* should have equal radius. This problem results in equations that have to be solved numerically in order to find the relationship between the points *L*, *M*, *N*.

Finally *L*, *M*, *N* are collinear if, and only if, *P* lies on the circumcircle of *ABC* and then *AX*, *BY*, *CZ* are concurrent, so that triangles *XYZ* and *ABC* are not only similar but in perspective. It follows from the work of Wood [1] on similar in perspective triangles that the perspector *Q* lies on both the circumcircles of *ABC* and *XYZ*. The circle *XYZ* may be identified as the circle of centres in the Wood configuration, see Bradley and Smith [2]. It therefore contains *O*, the circumcentre of *ABC*.

In Fig. 5.1 we show the Miquel configuration for an arbitrarily chosen point *P*, when the angle of rotation of the similarity is 30°.

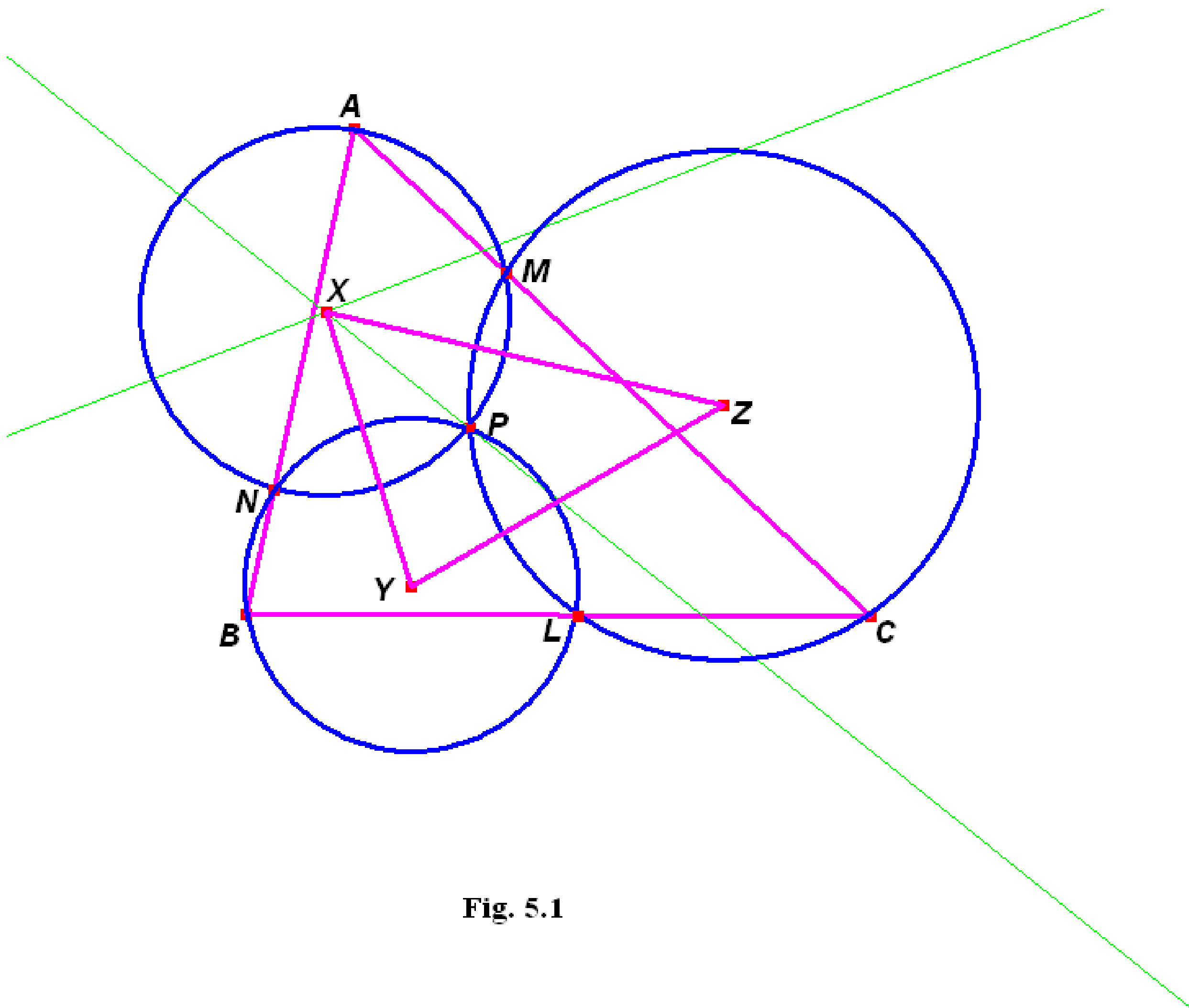

Fig. 5.1

2. Figure arising from arbitrary *L*, *M*, *N* on the sides and the consequential similarity

We start with a general, but economically parameterised triangle *ABC*, with $A(0, 0)$, $B(2, 2v)$, $C(2, 2w)$, $(w > v)$. The points *L*, *M*, *N* are now chosen on the sides *BC*, *CA*, *AB* respectively and to these are assigned co-ordinates $L(2, 2u)$, $M(h, hw)$, $N(k, kv)$ $(u \neq v, w;\ h, k \neq 0, 1)$.

The equations of the circles *BNL*, *CLM*, *AMN* may be obtained using standard methods involving 4 x 4 determinants and are respectively

$$x^2 + y^2 + (2(uv - 1) - k(v^2 + 1))x - 2(u + v)y + 2k(v^2 + 1) = 0, \tag{2.1}$$

$$x^2 + y^2 + (2(uw - 1) - h(w^2 + 1))x - 2(u + w)y + 2h(w^2 + 1) = 0, \tag{2.2}$$

$$(v - w)(x^2 + y^2) + (kw(v^2 + 1) - hv(w^2 + 1))x + (h(w^2 + 1) - k(v^2 + 1))y = 0. \tag{2.3}$$

It may be checked that these three circles share a common point *P*. We do not record its co-ordinates as they are complicated and are not needed in what follows. The co-ordinates of the centres of the three circles can now be determined and are

X : $\{1/(2(v-w))\}(hv(w^2+1)-kw(v^2+1), k(v^2+1)-h(w^2+1))$,
Y : $(½(k(v^2+1)-2(uv-1)), u+v)$,
Z : $(½(hw^2+1)-2(uw-1)), u+w)$.

Calculations may now be carried out to show that $XY/YZ = AB/BC = \{\sqrt{(v^2+1)}\}/(w-v)$, and $XZ/XY = AC/AB = \sqrt{\{(w^2+1)/(v^2+1)\}}$ and this establishes the similarity between triangle XYZ and ABC.

We now consider the problem of when the three Miquel circles have a common area, which is when $AL^2 = BM^2 = CN^2$. We find

$$AL^2 = (v^2+1)(w^2+1)(h^2(w^2+1)-2hk(vw+1)+k^2(v^2+1))/\{4(v-w)^2\}, \tag{2.4}$$

$$BM^2 = ¼\{k^2(v^2+1)^2-4k(v^2+1)(uv+1)+4(u^2+1)(v^2+1)\}, \tag{2.5}$$

$$CN^2 = ¼\{h^2(w^2+1)^2-4h(w^2+1)(uw+1)+4(u^2+1)(w^2+1)\}. \tag{2.6}$$

Solving for h, k in terms of u we find the condition for equal areas is

$$h = (2uv-v^2+1)/(vw+1), \tag{2.7}$$

and

$$k = (2uw-w^2+1)/(vw+1). \tag{2.8}$$

l

These formulas give the unique positions for M, N when L is given. A particularly obvious case is when $h = k = 1$ and $u = ½(v + w)$, which is when P is the circumcentre and L, M, N the midpoints of the sides. No other case exists in which $h = k$.

3. Figure arising from choosing an arbitrary position for *P* and constructing a direct similarity with centre *P*

In order to investigate the above direct similarity more fully we now investigate how a Miquel configuration can be constructed by choosing a point P to serve as a Miquel point and creating a direct similarity centre P for which the image of triangle ABC is a triangle XYZ so that X, Y, Z have the property that they are the centres of circles $APMN$, $BPNL$, $CPLM$ where L, M, N lie on BC, CA, AC respectively.

It is convenient, since rotation and dilation about P is to be effected, to choose P to be the origin, which means altering the co-ordinates of points. Now it is A, B, C and P that are fixed and the similarity that is specified. What happens is this. Triangle ABC is rotated about P by an angle θ to give a triangle $A'B'C'$. Now dilation is carried out by a scale factor d that has to be adjusted so that X lies on $A'P$ and is also on the perpendicular bisector of AP. This is necessary for X to be the centre of a circle through A and P. Y and Z are then fixed by the direct similarity and they are then centres of circles through B and P and through C and P respectively. It turns out that although any angle θ ($\theta \neq ½\pi$) can be chosen, X, Y, Z do not lie on the required perpendicular bisectors unless $d = 1/(2\cos\theta)$. Rotation through a right angle is forbidden in order that X, Y, Z should be finite points.

We choose the circumcircle of ABC to have radius 1 and equation

$$(x - k)^2 + y^2 = 1. \tag{3.1}$$

Then the circumcentre has co-ordinates $(k, 0)$ and P has co-ordinates $(0, 0)$. Points on the circle can be assigned a real parameter 't' so that they have co-ordinates $\{1/(1 + t^2)\}(t^2(k - 1) + (k + 1), 2t)$ and to keep matters sufficiently general we choose A, B, C to have parameters a, b, c respectively.

As indicated we now perform the direct similarity with rotation θ and scale factor $d = 1/(2\cos\theta)$, so that the matrix that represents it has rows $(½ , - s/(1 - s^2))$ and $(s/(1 - s^2), ½)$, where $s = \tan½\theta$.

The resulting co-ordinates of X are $\{(1/(2(1 - s^2)(1 + a^2)\}(a^2(k - 1)(1 - s^2) - 4as + (k + 1)(1 - s^2), 2(a^2s(k - 1) + a(1 - s^2) + s(k + 1)))$. Those of Y, Z may be obtained by replacing a by b, c respectively. It may now be checked that $XA = XP$, $YB = YP$, $ZC = ZP$. Circles centres X, Y, Z through P now form a Miquel configuration in which pairs of circles meet at points (other than P) on the sides of the triangle.

4. What happens when *P* lies on the circumcircle of *ABC*

What happens when P lies on the circumcircle is that the previous analysis may be used with $k = 1$. A now has co-ordinates $\{2/(a^2 + 1)\}(1, a)$ with similar expressions for B and C. The circumcircle ABC has equation

$$x^2 + y^2 - 2x = 0. \tag{4.1}$$

X has co-ordinates $\{1/((a^2 + 1)(s^2 - 1))\}(2as + s^2 - 1, a(s^2 - 1) - 2s)$ with similar expressions for Y and Z. The circle centre X passing through A has equation

$$(a^2 + 1)(s^2 - 1)(x^2 + y^2) - 2(2as + s^2 - 1)x - 2(a(s^2 - 1) - 2s)y = 0. \tag{4.2}$$

The circle centre Y passing through B has a similar equation with b replacing a. These two circles meet at P and at the point N with co-ordinates $\{1/((s^2 - 1)(a^2 + 1)(b^2 + 1))\}(2(ab - 1)(1 - s^2) + 2s(a + b), 2(2s(ab - 1) + (a + b)(s^2 - 1))$, with similar expressions for L and M. It is straightforward to show these points lie on the sides of ABC. Forming the 3x 3 determinant whose rows are the co-ordinates of L, M, N (with third element 1) we find that it vanishes, showing that LMN is a straight line.

The equation of AX is

$$(a(s^2 - 1) + 2s)x + (2as - (s^2 - 1))y - 4s = 0. \quad (4.3)$$

This line meets the circumcircle with Equation (3.1) at the point *Q* with co-ordinates $\{(4s/(s^2 + 1)^2\}(2s, 1 - s^2)$. As this is independent of *a*, lines *BY*, *CZ* also pass through *Q*, which means that triangles *ABC* and *XYZ* are also in perspective.

Circle *XYZ* has equation

$$(s^2 - 1)(x^2 + y^2) + (1 - s^2)x + 2sy = 0. \quad (4.4)$$

This circle passes through *P* and also through *O*, the centre of the circumcircle of *ABC*, showing that circle *XYZ* is the circle of centres in the Wood [1] configuration based on *ABCPQ*. It may also be shown that *LMN* is the double Wallace-Simson line of *P* with respect to triangle *XYZ* triangle. Since the construction is reversible this demonstrates that every transversal of *ABC* is the Double Simson line of a point *P* on its circumcircle with respect to some triangle *XYZ* that is directly similar to *ABC* with centre of similarity *P*. See Fig. 2 for a record of these results.

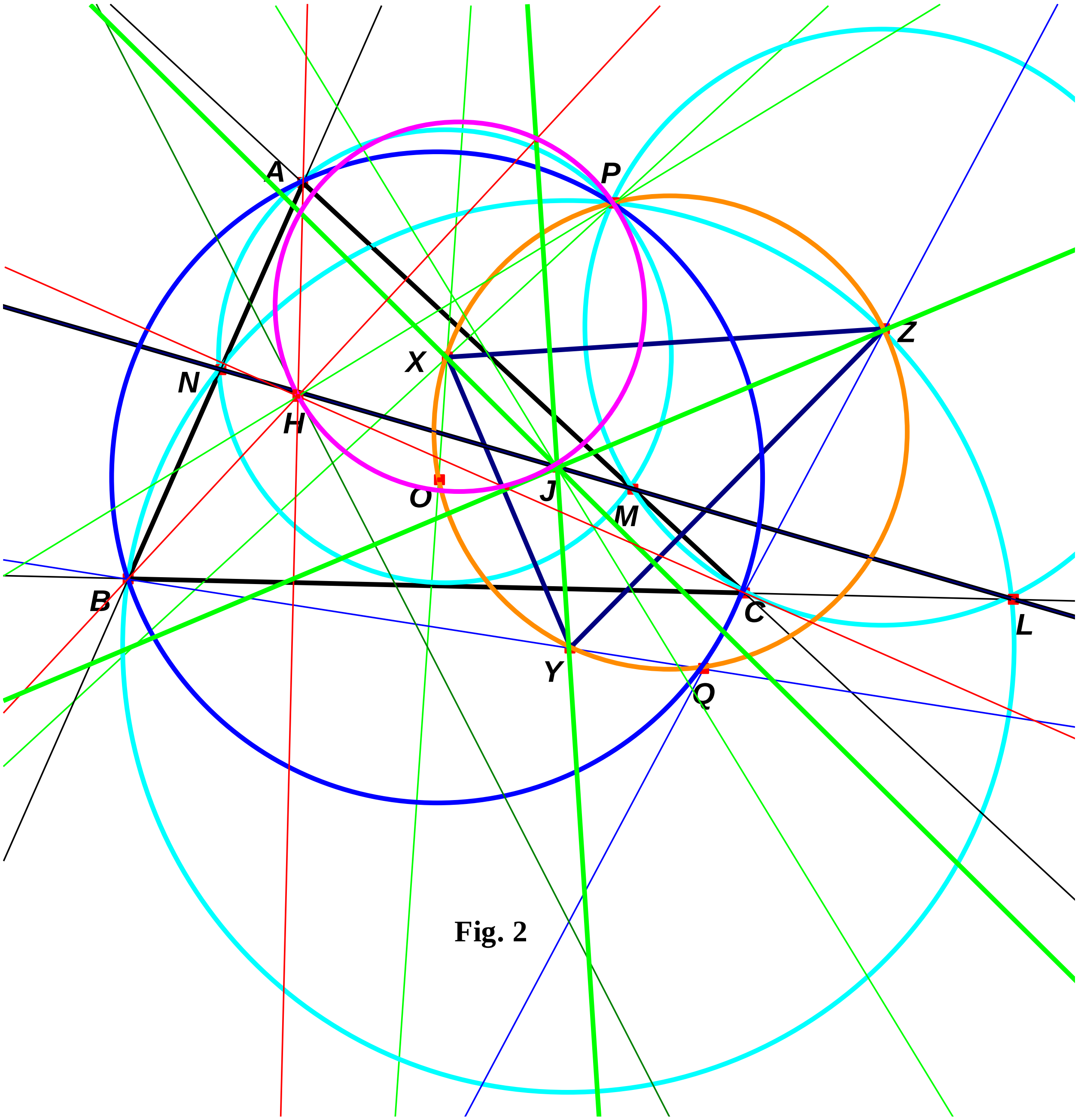

Fig. 2

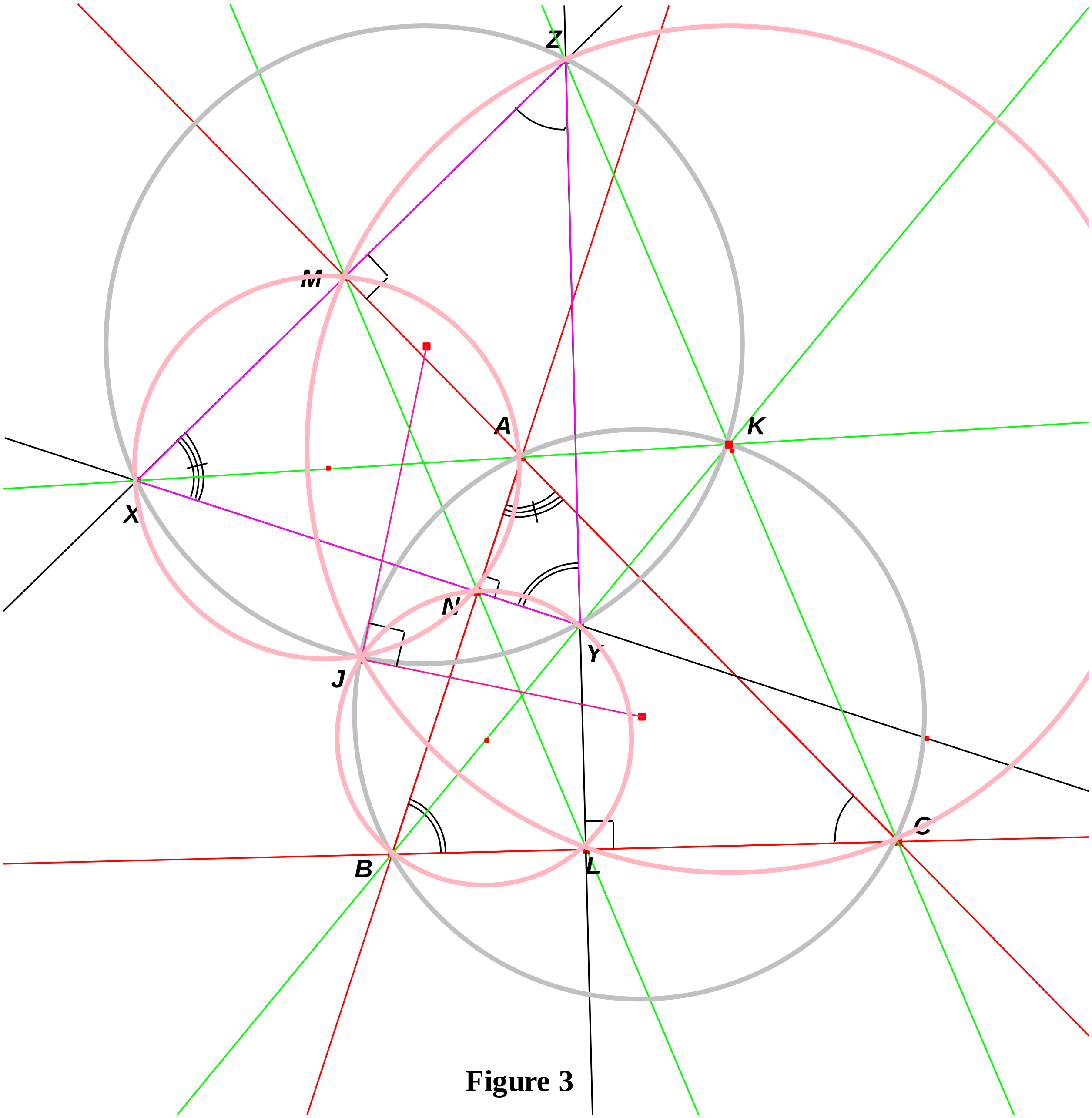

Figure 3

5. Paralogic Wood configuration

This is a very natural construction. Start with a triangle ABC and a transversal LMN, with L on BC etc. Now erect perpendiculars at L, M, N to BC, CA, AB respectively to create by their intersections in pairs the triangle XYZ.

The ABC and XYZ are directly similar and in perspective with LMN the Desargues' axis and the circles ABC and XYZ meet at the perspector K and the centre of inverse symmetry J. Moreover the circles are orthogonal, so the triangles ABC and XYZ are paralogic. The paralogic centres are the orthocentres of the two triangles. See Figure 3.

Reference

1. F.E. Wood, *Amer. Math. Monthly* 36:2 (1929) 67-73.

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