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A Twelve Point Configuration and Carnot's Theorem

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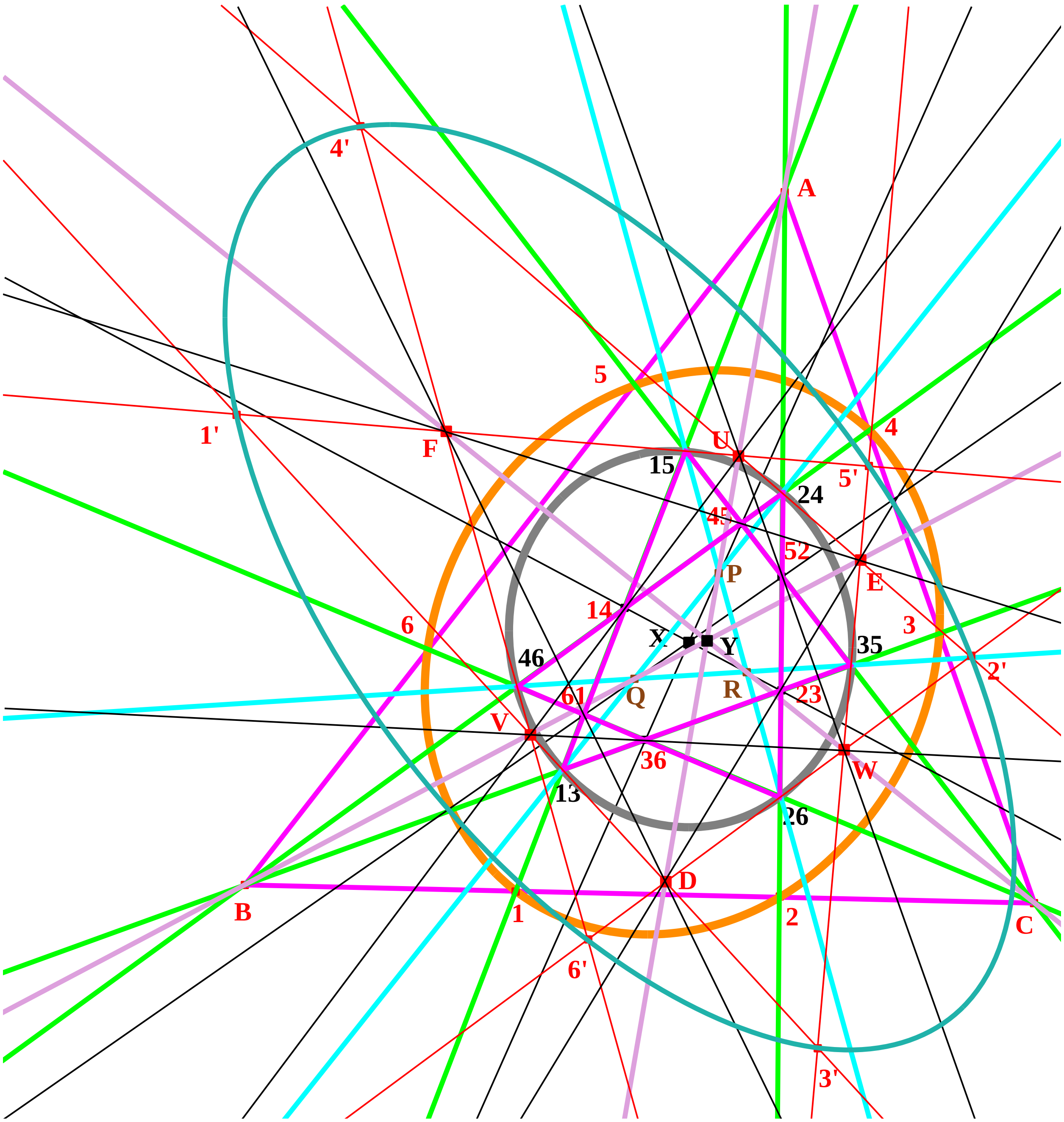

Fig. 1

A conic intersecting all three sides of a triangle gives rise to a wonderful configuration

Abstract: A twelve point configuration occurs when six of the points lie on a conic and the remaining six points are the vertices of two triangles in perspective. In this article we consider the configuration in which a conic cuts the three sides of a triangle in real points. A number of theorems create a variety of conditions on the points of intersection equivalent to Carnot's theorem.

1. Introduction

The construction of Fig.1 proceeds as follows: Given a conic Σ cutting BC at points 1, 2, CA at 3, 4 and AB at 5, 6 then determine the points of intersection such as 13, the intersection of A1 and B3. Altogether there are twelve such points 13, 26, 35, 24, 15, 46, 61, 36, 23, 52, 45, 14. We establish the result that the first six lie on a conic Γ and the lines 61 52, 36 45 and 23 14 are concurrent at a point we label X. But that is not the end of the matter; four other points of concurrence exist. There is P = 36 45^15 26^24 13, Q = 61 52^35 46^ 13 24, R = 14 23^15 26^35 46 and finally Y = AP^BQ^CR. We establish these results and their converses, showing them to be all equivalent to Carnot's theorem that

$$(B1/C1)(B2/C2)(C3/A3)(C4/A4)(A5/B5)(A6/B6) = 1 \quad (1.1)$$

It appears that the configuration can be developed further (to an infinite extent). For example the tangent to Γ at 15 meets AP at U and CY at F, with V, W and D, E similarly defined. The F also lies on the tangent at 46 and U lies on the tangent at 24 and so on. Then the tangents at 15 and 13 meet at a point 1', the tangents at 46 and 26 meet at 6' and so on and then 1', 2', 3', 4', 5', 6' are co-conic. We do not establish these latter results analytically, as the article is long enough as it is, but the results are confirmed by *CABRI II plus*. See Fig. 1.

2. Points 1, 2, 3, 4, 5, 6 and the conic Σ

We take points 1, 3, 5 to have co-ordinates 1(0, l, 1 –l), 3(1 – m, 0, m), 5(n, 1 – n, 0) and points 2, 4, 6 to have co-ordinates 2(0, 1 – p, p), 4(q, 0, 1 – q), 6(1 – r, r, 0). The conic Σ has an equation of the form

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \quad (2.1)$$

and putting in the conditions that points 1, 2, 3, 4, 5 lie on Σ we obtain

$$u = 2lmn(n - 1)(p - 1)(q - 1), \quad (2.2)$$
$$v = 2npq(l - 1)(m - 1)(n - 1), \quad (2.3)$$
$$w = 2nlq(m - 1)(n - 1)(p - 1), \quad (2.4)$$
$$f = nq(m - 1)(n - 1)(2lp - l - p + 1), \quad (2.5)$$
$$g = nl(n - 1)(p - 1)(2mq - m - q + 1), \quad (2.6)$$
$$h = l(m(n^2(2pq - p - q + 1) - 2npq + pq) - pq(n - 1)^2) - pq(m - 1)(n - 1)^2. \quad (2.7)$$

After some heavy algebra we find the condition that point 6 lies on Σ is that

$$r/(1 - r) = \{lmn(1 - q)(1 - p)\}/\{pq(1 - l)(1 - m)(1 - n)\}. \quad (2.8)$$

In fact what we have now done is to verify Carnot's theorem (1.1).

3. The six points of intersection 13, 26, 35, 24, 15, 46 and the conic on which they lie

The lines A1, B3, C5 have equations $(1 - l)y = lz$, $(1 - m)z = mx$, $(1 - n)x = ny$ respectively. The lines A2, B4, C6 have equations $py = (1 - p)z$, $qz = (1 - q)x$, $rx = (1 - r)y$ respectively. From equation (2.8) the equation of C6 is now taken as

$$lmn(1 - p)(1 - q)x = pq(1 - l)(1 - m)(1 - n)y. \quad (3.1)$$

We now give the co-ordinates of six of the points of intersection of these lines:

13 = A1^B3: $x = (1 - l)(1 - m)$, $y = lm$, $z = m(1 - l)$. (3.2)

35 = B3^C5: $x = n(1 - m)$, $y = (1 - m)(1 - n)$, $z = mn$. (3.3)

15 = A1^C5: $x = nl$, $y = l(1 - n)$, $z = (1 - n)(1 - l)$. (3.4)

26 = A2^C6: $x = (1 - r)(1 - p)$, $y = (1 - p)r$, $z = rp$. (3.5)

24 = A2^B4: $x = pq$, $y = (1 - p)(1 - q)$, $z = p(1 - q)$. (3.6)

46 = B4^C6: $x = q(1 - r)$, $y = qr$, $z = (1 - q)(1 - r)$. (3.7)

It may now be shown that these six points lie on a conic if, and only if, Equation (2.8) holds and then the conic has the form (2.1) with

$$u = 2lmn(1 - n)(1 - q)^2(l + p - 1)(m(n(2p - 1) - p) + p(1 - n)), \quad (3.8)$$

$$v = 2np^2(1 - l)(1 - m)(1 - n)(m + q - 1)(l(n(2q - 1) - q) + q(1 - n)), \quad (3.9)$$

$$w = 2l(m - 1)(l(m(n^2(p + q - 1) - 2npq + pq) - pq(n - 1)^2) - pq(m - 1)(n - 1)^2)$$
$$\text{all times } (n(p(2q - 1) - q + 1) - pq), \quad (3.10)$$

$$f = p(m - 1)(l^2(m + n - 1)(2qn - n - q) + l(n - 1)(m(n(2q - 1) - 2q) + n(1 - 3q) + 2q)$$
$$- q(m - 1)(n - 1)^2(pq - n(p(2q - 1) - q + 1)), \quad (3.11)$$

$$g = l(1 - q)(l(m + n - 1) + (1 - m)(1 - n))(m(n(2p - 1) - p) - p(n - 1))$$
$$\text{all times } (n(p(2q - 1) - q + 1) - pq), \quad (3.12)$$

$$h = p(1 - q)(l^2(m + n - 1)(2nq - n - q) + l(n - 1)(m(n(2q - 1) - 2q) + n(1 - 3q) + 2q)$$
$$- q(m - 1)(n - 1)^2(m(n(2p - 1) - p - p(n - 1)). \quad (3.13)$$

4. The six points of intersection 61, 36, 23, 52, 45, 14 and the point X

We now give the co-ordinates of the other six points of intersection:

14 = A1^B4: $x = q(1 - l)$, $y = l(1 - q)$, $z = (1 - l)(1 - q)$, (4.1)

61 = C6^A1: $x = l(1 - r)$, $y = lr$, $z = r(1 - l)$, (4.2)

36 = B3^C6: $x = (1 - m)(1 - r)$, $y = r(1 - m)$, $z = m(1 - r)$, (4.3)

23 = A2^B3: $x = p(1 - m)$, $y = m(1 - p)$, $z = mp$, (4.4)

52 = C5^B2: $x = n(1 - p), y = (1 - n)(1 - p), z = p(1 - n),$ (4.5)
45 = B4^C5: $x = nq, y = q(1 - n), z = n(1 - q).$ (4.6)

The equation of 23 14 is

$$m(l + p - 1)(q - 1)x + p(1 - l)(m + q - 1)y - (l(m(p - q) + p(q - 1)) + mq(1 - p)) = 0. \quad (4.7)$$

The equation of 36 45 is

$$(m(n(q - r) + q(r - 1)) + nr(1 - q))x + n(1 - r)(m + q - 1)y + q(m - 1)(n + r - 1)z = 0. \quad (4.8)$$

The equation of 61 52 is

$$r(1 - n)(l + p - 1)x +(l(n(p - r) + p(r - 1)) + nr(1 - p))y + l(n + r - 1)(p - 1)z = 0. \quad (4.9)$$

It may now be shown that these three lines are concurrent at a point X if, and only if, equation (2.8) holds. The co-ordinates of X are (x, y, z), where (with r given by (2.8)),

$$x = - l(m(n(p(q + r - 1) - qr - r + 1) + pq(r - 1)) + n(p(q(r - 2) - r + 1) + q + r - 1) + pq(1 - r)) + mnqr(p - 1) + nqr(1 - p), \quad (4.10)$$

$$y = l(r(n(p(q - 1) + 1) - q) - m(n(p(q - r) + q(r - 1) + r) + q(p(r - 1) - 2r + 1))) + qr(1 - p)(1 - m)(1 - n), \quad (4.11)$$

$$z = l(m(n(p(q - r) - r(q - 1)) + pq(r - 1)) + nr(1 - q)(p - 1)) + nr(1 - p)(1 - q)(1 - m). \quad (4.12)$$

5. Lines 24 13, 36 45, 15 26, are concurrent at P and Points Q, R and Y

The equation of the line 24 13 is

$$m(l + p - 1)(q - 1)x + p(l - 1)(m + q - 1)y + (l(m(p + q - 1) + (p - 1)(q - 1)) + (m - 1)(p - 1)(q - 1)) = 0. \quad (5.1)$$

The equation of the line 36 45 is

$$(m(n(q - r) + q(r - 1)) + nr(1 - q))x + n(1 - r)(m + q - 1)y + q(m - 1)(n + r - 1)z = 0. \quad (5.2)$$

The equation of the line 15 26 is

$$r(1 - n)(l + p - 1)x - (l(n(p + r - 1) + (p - 1)(r - 1)) + (n - 1)(p - 1)(r - 1))y + l(1 - p)(n + r - 1)z = 0. \quad (5.3)$$

It may now be verified that these three lines are concurrent at a point P if, and only if, equation (2.8) holds. The co-ordinates of P (with r given by (2.8)) are (x, y, z), where

$$x = - l(m(n(p(q + r - 1) + q(r - 1)) + q(p - 1)(r - 1)) + n(p(q(r - 2) - r + 1) + (1 - r)(2q - 1)) + q(1 - r)(p - 1)) + mq(1 - r)(1 - n)(1 - p) + q(1 - n)(1 - p)(1 - r), \quad (5.4)$$

$$y = l(r(n(p(q - 1) - 2q + 1) + q) - m(n((p(q - r) - q(r + 1) + r) + q(p(r - 1) + 1)))$$
$$+ qr(m - 1)(n - 1)(p - 1), \quad (5.5)$$
$$z = l(m(n(p(q - r) + q(r - 1)) + q(p - 1)(r - 1)) + npr(1 - q)) + mq(n - 1)(p - 1)(r - 1). \quad (5.6)$$

It now follows by symmetry that lines 13 24, 46 35, 61 52 are concurrent at a point Q and that lines 15 26, 35 46, 14 23 are concurrent at a point R.

Finally it may be shown that AP, BQ, CR meet at a point Y with co-ordinates (x, y, z), where

$$x = \{p(m - 1)\}/\{m(n - p) - p(n - 1)\}, \quad (5.7)$$
$$y = \{l(q - 1)\}/\{l(n - q) - q(n - 1)\}, \quad (5.8)$$
$$z = \{p(1 - q)\}/\{n(p + q - 1) - pq\}. \quad (5.9)$$

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