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Generating Circles from the Symmedian point

Christopher Bradley

Abstract: Given a triangle ABC and its symmedian point K circles BKC, CKA, AKB are drawn. Their other points of intersection with the sides of ABC are shown to lie on a Tucker circle. These points also lie in pairs on the sides of another triangle and the other intersections of these sides with the circles BKC, CKA, AKB are also concyclic. Centres of these circles are shown to lie on the Brocard axis.

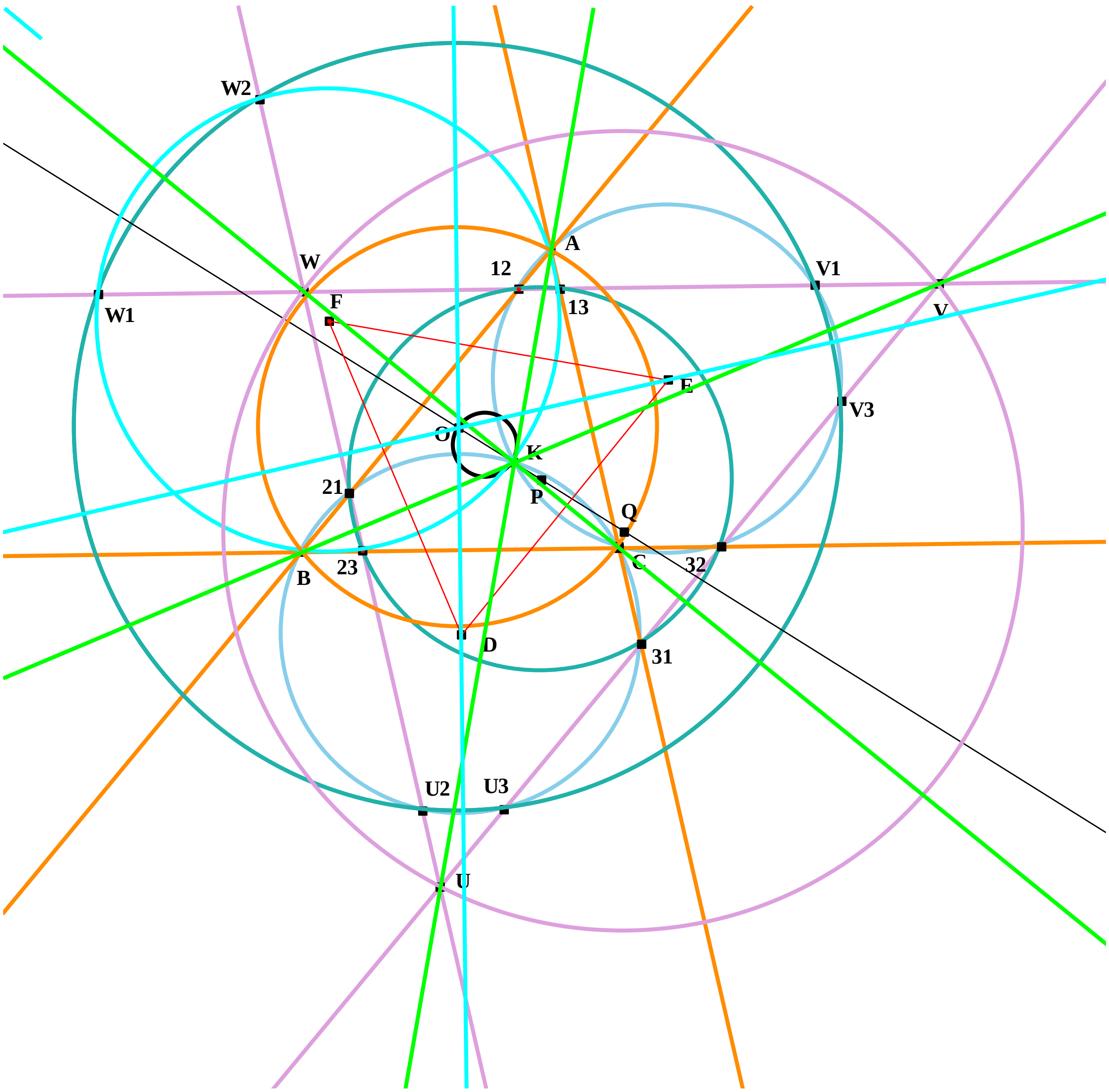

Fig. 1

1. Introduction

The construction of Fig. 1 is reasonably straightforward. Given triangle ABC, symmedian point K and circumcentre O, the circles BKC, CKA, AKB centres D, E, F respectively are drawn. The second intersections of circle BKC with AB and AC are 21 and 31 respectively. The second intersections of circle CKA with AB and BC are 12 and 32 respectively. The second intersections of circle AKB with AC and BC are 13 and 23 respectively. It is proved that these six points lie on a circle in which 12 13 is parallel to BC etc. and hence is a Tucker circle. Its centre P is shown to lie on the Brocard axis OK.

When lines 12 13, 23 21 and 31 32 are extended a triangle UVW is formed whose centre Q is also shown to lie on the Brocard axis. When the sides of UVW are extended the second intersections of VW with circles CKA and AKB are points V_1, W_1 respectively. Similarly W_2 and U_2 lie on WU and U_3 and V_3 lie on UV.

It is shown that these six points also lie on a circle, this time with centre O, the circumcentre of ABC. The construction can clearly be continued indefinitely creating an endless chain of circles.

Areal co-ordinates, with ABC triangle of reference are used in following sections.

2. Circles BKC, CKA, AKB and their centres D, E, F

Circles in areal co-ordinates have equations of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0. \qquad (2.1)$$

The point K has co-ordinates (a^2, b^2, c^2).

Circle BKC has v = w = 0 and

$$u = -\,3b^2c^2/(a^2 + b^2 + c^2). \qquad (2.2)$$

Circle CKA has w = u = 0 and

$$v = -\,3c^2a^2/(a^2 + b^2 + c^2). \qquad (2.3)$$

Circle AKB has u = v = 0 and

$$w = -\,3a^2b^2/(a^2 + b^2 + c^2). \qquad (2.4)$$

The centre of a circle written in the form of equation (2.1) has centre (x, y, z), where

$$x = -\,(a^4 - a^2(b^2 + c^2 + 2u - v - w) + (b^2 - c^2)(v - w)), \qquad (2.5)$$

and y, z follow by cyclic change of a, b, c and u, v, w.

The centre D therefore has co-ordinates (x, y, z), where

$$x = a^2(b^4 + c^4 – a^4 – 4b^2c^2), \quad (2.6)$$

$$y = b^2(a^4 – b^4 – 2c^4 + 3b^2c^2 + 5c^2a^2), \quad (2.7)$$

$$z = c^2(a^4 – 2b^4 – c^4 + 3b^2c^2 + 5a^2b^2). \quad (2.8)$$

The co-ordinates of the centres E and F may be written down from Equations (2.6) – (2.8) by cyclic change of x, y, z and a, b, c.

3. The six points 31, 21, 12, 32, 23, 13 lie on a circle centre P

Circle BKC meets CA (y = 0) at the point 31 with co-ordinates $(a^2 + b^2 – 2c^2, 0, 3c^2)$.
Circle BKC meets AB (z = 0) at the point 21 with co-ordinates $(c^2 + a^2 – 2b^2, 3b^2, 0)$.
Circle CKA meets AB (z = 0) at the point 12 with co-ordinates $(3a^2, b^2 + c^2 – 2a^2, 0)$.
Circle CKA meets BC (x = 0) at the point 32 with co-ordinates $(0, a^2 + b^2 – 2c^2, 3c^2)$.
Circle AKB meets BC (x = 0) at the point 23 with co-ordinates $(0, 3b^2, c^2 + a^2 – 2b^2)$.
Circle AKB meets CA (y = 0) at the point 13 with co-ordinates $(3a^2, 0, b^2 + c^2 – 2a^2)$.

It may now be checked that the circle through these six points has an equation of the form (2.1) with

$$u = 3b^2c^2(2a^2 – b^2 – c^2)/(a^2 + b^2 + c^2)^2, \quad (3.1)$$

$$v = 3c^2a^2(2b^2 – c^2 – a^2)/(a^2 + b^2 + c^2)^2, \quad (3.2)$$

$$w = 3a^2b^2(2c^2 – a^2 – b^2)/(a^2 + b^2 + c^2)^2. \quad (3.3)$$

Using Equation (2.5) we find the x-co-ordinate of its centre P to be

$$x = a^2(a^4 + 2(b^4 + c^4) – 2b^2c^2 – 3a^2(b^2 + c^2)). \quad (3.4)$$

Its y- and z- co-ordinates follow by cyclic change of a, b, c.

It may now be checked that P lies on the Brocard axis with equation

$$b^2c^2(b^2 – c^2)x + c^2a^2(c^2 – a^2)y + a^2b^2(a^2 – b^2)z = 0. \quad (3.5)$$

4. The triangle UVW and the circle UVW centre Q

The equation of the line 12 13 is

$$(2a^2 – b^2 – c^2)x + 3a^2(y + z) = 0. \quad (4.1)$$

This is clearly parallel to BC and hence the circle of Section 3 is a Tucker circle and also it meets the line 21 23 at W with co-ordinates $(3a^2, 3b^2, c^2 – 2a^2 – 2b^2)$. U and V have co-ordinates that may be obtained from those of W by cyclic change of x, y, z and a, b, c.

Circle UVW has equation of the form (2.1) with

$$u = 3b^2c^2(7a^2 – 2(b^2 + c^2))/(a^2 + b^2 + c^2)^2, \qquad (4.2)$$

with v, w obtained from Equation (4.2) by cyclic change of a, b, c.

Using Equation (2.5) we find the x-co-ordinate of its centre Q to be

$$x = a^2(a^4 + 5b^4 + 5c^4 – 2b^2c^2 – 6a^2(b^2 + c^2)), \qquad (4.3)$$

with the y- and z- co-ordinates obtained from (4.3) by cyclic change of a, b, c.

It may now be checked that Q lies on the Brocard axis with Equation (3.5).

It is also the case that AU, BV, CW obviously concur at K.

5. Points W_1, V_1, U_2, W_2, V_3, U_3 and the circle through these six points

The line 12 13 meets circle AKB at the point W_1 with co-ordinates $(3a^2, b^2 + 4c^2 – 2a^2, – 3c^2)$.
The line 12 13 meets circle CKA at the point V_1 with co-ordinates $(3a^2, – 3b^2, c^2 + 4b^2 – 2a^2)$.
Similarly U_2 has co-ordinates $(– 3a^2, 3b^2, c^2 +4a^2 – 2b^2)$, W_2 has co-ordinates $(a^2 + 4c^2 – 2b^2, 3b^2, – 3c^2)$, V_3 has co-ordinates $(a^2 + 4b^2 – c^2, – 3b^2, 3c^2)$ and U_3 has co-ordinates $(– 3a^2, b^2 + 4a^2 – 2c^2, 3c^2)$.

It may now be shown that these six points lie on a circle of the form (2.1) with

$$u = v = w = 9a^2b^2c^2/(a^2 + b^2 + c^2)^2. \qquad (5.1)$$

As with all circles with u = v = w its centre is the circumcentre O.

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.