Article 15

Four Concurrent Euler Lines

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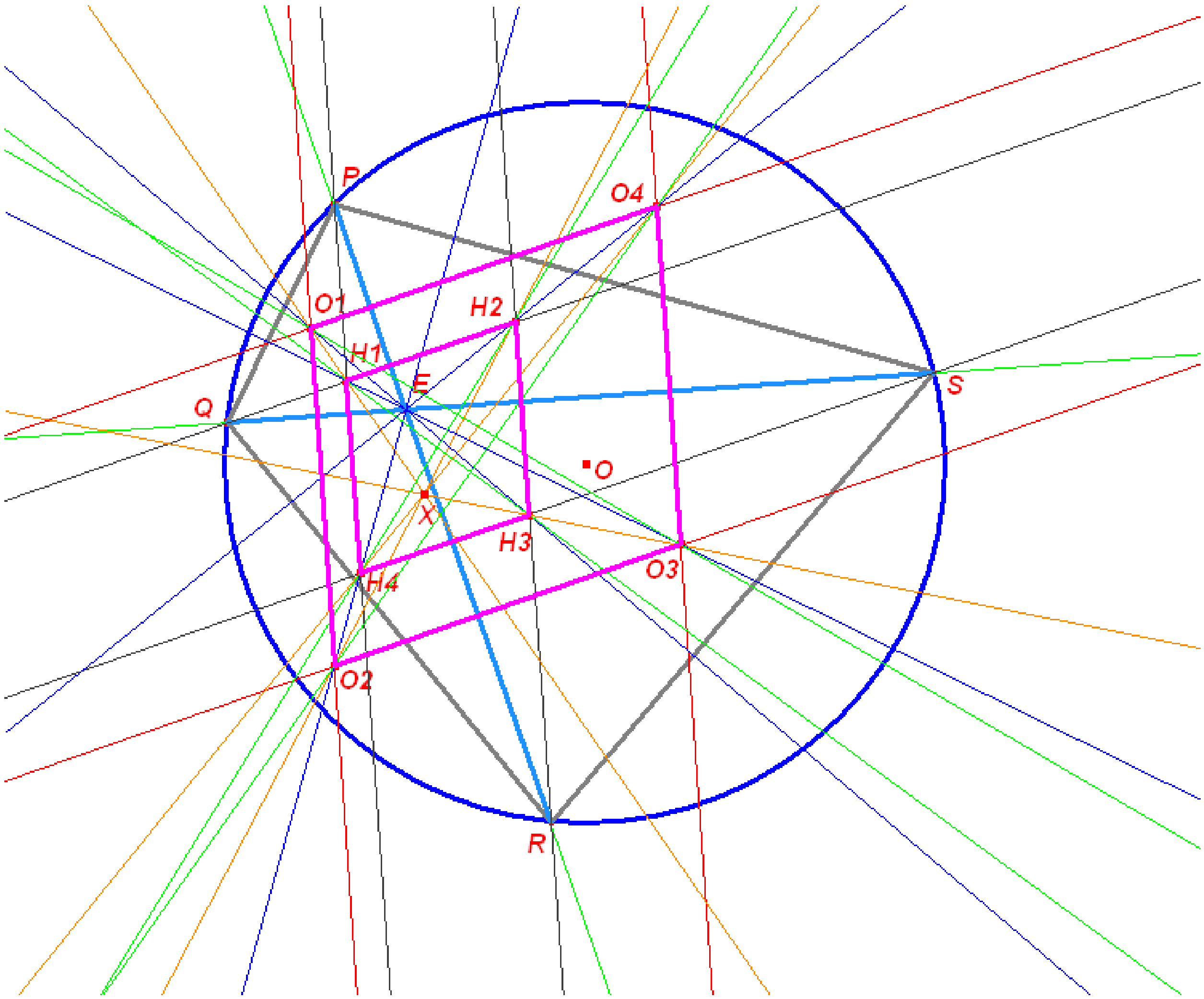
1. Introduction

In this paper we consider a cyclic quadrilateral *PQRS* in which the diagonals *PR* and *QS* meet at a point *E* and we prove a number of results about the triangles *PQE*, *QRE*, *RSE*, *SPE*.



These are:

1. The Euler lines of the four triangles are concurrent at a point *X*;

2. If the circumcentres of the four triangles are denoted by O_1, O_2, O_3, O_4 respectively then $O_1O_2O_3O_4$ is a parallelogram;
3. If the orthocentres of the four triangles are denoted by H_1, H_2, H_3, H_4 respectively then $H_1H_2H_3H_4$ is a parallelogram;
4. The parallelograms $O_1O_2O_3O_4$ and $H_1H_2H_3H_4$ are inversely similar, with O_1O_2 parallel to H_1H_4 etc.;
5. O_1H_3, O_3H_1, O_2H_4, O_4H_2 are concurrent at E.

Results 2 and 3 are known results, see Bradley [1], but they are inevitably given explicit proof in the course of the calculations required to prove the other results. The results are illustrated in the figure. It is also the case that the analysis takes no account of the order of the vertices P, Q, R, S and is equally valid when P and R are adjacent vertices. This means that Results 1-5 are also true for the cyclic quadrilateral $PQRS$ when E is replaced by $F = PQ$^RS or $G = QR$^SP, with the four triangles replaced in an obvious way. Analysis is carried out using Cartesian co-ordinates. Singular cases can occur when one or other of the key points recede to infinity. We assume in our analysis that all points arising are finite and distinct. Calculations were carried out using the algebra package *DERIVE* and have been subject to various consistency tests.

2. Preliminary results

We consider a triangle ACE, with $E(0, 0)$, $A(a, b)$, $C(c, d)$, $(ad \neq bc)$, and work out the co-ordinates of its circumcentre O, its orthocentre H and the equation of its Euler line OH. In subsequent sections we identify the vertices A, C successively as the pairs P, Q; Q, R; R, S; S, P. The equations of EA, EC, AC are immediate and are respectively

$$ay = bx, \tag{2.1}$$

$$cy = dx \tag{2.2}$$

and

$$x(b - d) + y(c - a) + ad - bc = 0 \tag{2.3}$$

The perpendicular bisectors of EA and EC are respectively

$$2(ax + by) = a^2 + b^2 \tag{2.4}$$

and

$$2(cx + dy) = c^2 + d^2. \tag{2.5}$$

These meet at the circumcentre O with co-ordinates (x, y), where

$$x = \frac{(a^2 + b^2)d - (c^2 + d^2)b}{2(ad - bc)} \tag{2.6}$$

$$y = \frac{(a^2 + b^2)c - (c^2 + d^2)a}{2(bc - ad)}\,. \tag{2.7}$$

It may be checked that this also lies on the perpendicular bisector of the line AC. The three altitudes have equations

$$y(b - d) = x(c - a), \tag{2.8}$$
$$ax + by = ac + bd, \tag{2.9}$$
$$cx + dy = ac + bd. \tag{2.10}$$

These meet at the orthocentre H with co-ordinates (x, y), where

$$x = \frac{(d - b)(ac + bd)}{ad - bc}, \tag{2.11}$$
$$y = \frac{(a - c)(ac + bd)}{ad - bc}. \tag{2.12}$$

We may now calculate the equation of the line OH, the Euler line of triangle ACE, and the result is

$$(3ac(a - c) + ad(2b - d) + bc(b - 2d))x + (3bd(b - d) + ad(a - 2c) + bc(2a - c))y - (a^2 + b^2 - c^2 - d^2)(ac + bd) = 0. \tag{2.13}$$

It may be checked that the centroid $\frac{1}{3}(a + c, b + d)$ lies on this line.

3. Co-ordinates of key points in the configuration

The ease with which a calculation may be carried out depends critically on the choice of the co-ordinates of E, P, Q, R, S. As in Section 2 the point E is taken as origin. For the points P, Q, R, S we take QS to be the x-axis and PR to be inclined to it in a direction determined by the unit vector $\left(\frac{(1-t^2)}{(1+t^2)}, \frac{2t}{(1+t^2)}\right)$ and use the converse of the intersecting chord theorem to specify their co-ordinates as $P(km(1 - t^2), 2kmt)$, $Q(-mn(1 + t^2), 0)$, $R(-nl(1 - t^2), -2nlt)$, $S(kl(1 + t^2), 0)$. With these co-ordinates the equation of the circle $PQRS$ is

$$2t(x^2 + y^2) + 2t(1 + t^2)(mn - kl)x + (1 + t^2)\{(nl - km)(1 + t^2) + (kl - mn)(1 - t^2)\}y - 2klmnt(1 + t^2)^2 = 0. \tag{3.1}$$

With E as origin then the analysis of Section 2 may be applied to each of the triangles PQE, QRE, RSE, SPE in turn by varying the values of a, b, c, d.

Result 1

The four Euler lines of these triangles are concurrent at the point X with co-ordinates (x, y), where

$$x = \frac{(1+t^2)(1-t^2)\{(nl-km)(1+t^2) + 2(kl-mn)(1-t^2)\}}{(3t^4 - 10t^2 + 3)}, \tag{3.2}$$
$$y = \frac{(1+t^2)(1-t^2)\{3(mk-nl)(1-t^4) + (mn-kl)(3t^4 - 2t^2 + 3)\}}{2t(3t^4 - 10t^2 + 3)}. \tag{3.3}$$

We now catalogue the co-ordinates of the points O_1, O_2, O_4 the circumcentres of triangles of triangles *PQE*, *QRE*, *SPE* and their corresponding orthocentres H_1, H_2, H_4. The co-ordinates of O_3, H_3 for triangle *RSE* are omitted as results involving these points follow by similar reasoning.

$O_1\,(-\tfrac{1}{2}mn(1+t^2),\ (1/4t)(m(1+t^2))\{k(1+t^2)+n(1-t^2)\})$;
$O_2\,(-\tfrac{1}{2}mn(1+t^2),\ (1/4t)(n(1+t^2))\{m(1-t^2)-l(1+t^2)\})$;
$O_4\,(\tfrac{1}{2}kl(1+t^2),\ (1/4t)(k(1+t^2))\{m(1+t^2)-l(1-t^2)\})$;
$H_1\,(km(1-t^2),\ -(1/2t)(m(1-t^2))\{k(1-t^2)+n(1+t^2)\})$;
$H_2\,(-ln(1-t^2),\ (1/2t)(n(1-t^2))\{l(1-t^2)-m(1+t^2)\})$;
$H_4\,(km(1-t^2),\ (1/2t)(k(1-t^2))\{l(1+t^2)-m(1-t^2)\})$.

4. The remaining results

Since Results 2 and 3 are known we do not provide further details, except to say that these results are a consequence of the fact that O_1O_2, O_3O_4 are perpendicular to *QS*, as are H_1H_4 and H_3H_2 all having infinite gradient, and the fact that the other four sides are perpendicular to *PR* having gradient $\frac{(t^2-1)}{2t}$.

Result 4

We are now able to compute the squares of various lengths and find that

$$O_1O_2^2 = (1+t^2)^4(km+ln)^2/16t^2\ ; \tag{4.1}$$

$$H_1H_2^2 = (1-t^4)^2(km+ln)^2/4t^2\ ; \tag{4.2}$$

$$O_1O_4^2 = (1+t^2)^4(kl+mn)^2/16t^2\ ; \tag{4.3}$$

$$H_1H_4^2 = (1-t^4)^2(kl+mn)^2/4t^2\ ; \tag{4.4}$$

From which we deduce

$$O_1O_2^2/H_1H_2^2 = O_1O_4^2/H_1H_4^2 = (1+t^2)^2/4(1-t^2)^2, \tag{4.5}$$

which, together with Results 2 and 3, establishes Result 4 that the two parallelograms are inversely similar. It is interesting to note that the enlargement factor of the similarity depends only on the angle between *PR* and *QS*.

Result 5

Inspection of the co-ordinates given at the end of Section 3 shows that $O_1O_2O_3O_4$ is also in perspective with $H_3H_4H_1H_2$ with vertex of perspective the origin *E*.

Reference

1. C. J. Bradley, *The Algebra of Geometry*, Bath: Highperception (2007).

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