Article: CJB/2011/149

Circles with a common point in a Cyclic Quadrilateral

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Abstract: If ABCD is a cyclic quadrilateral it is possible to find points P, Q, R, S, T, U on sides AB, BC, CD, DA, AC, BD respectively so that circles BPQU, APST, CQRT, DRSU have a common point M. These circles have centres that are concyclic, as are points M, U, I, T where I = AC^BD.

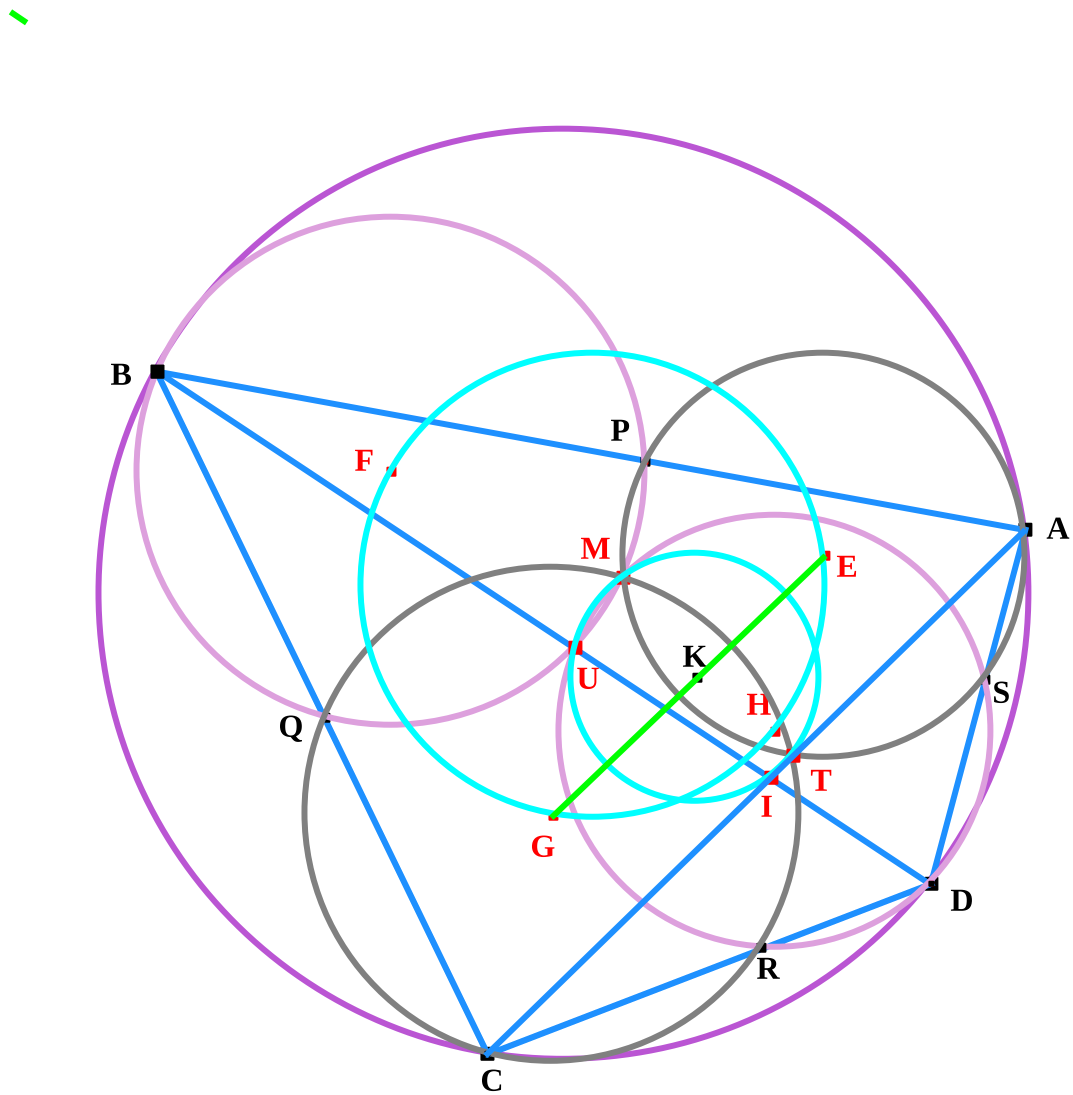

Fig. 1

A Common Miquel point M for triangles ABD, CBD, ABC, ADC in a cyclic quadrilateral

1. The construction

The important thing is that the above figure does not merely exist but can be constructed. There are, in fact, three degrees of freedom, which may be thought of as allowing free positioning of points P, Q and S. The construction then is as follows:

Choose P and Q and draw circle BPQ to meet BD at U. Choose S and draw circle SDU to meet CD at R and circle BPQU again at M. Then draw circle CRQ and discover that it passes through M and meets AC at T. Now draw circle APS and discover that it passes through M and T. Let AC and BD meet at I and discover that I, T, M, U are concyclic. Finally if E, F, G, H are the centres of circles APS, BPQ, CQR, DRS respectively then discover that E, F, G, H are concyclic.

Note that the results other than the concyclic properties (EFGH, ITMU) are **true for any quadrilateral**. The concyclic properties are true if ABCD is a cyclic quadrilateral.

In the following sections the above properties are proved analytically using areal co-ordinates. It is all technically rather difficult and so certain results are left for the interested reader to check (using algebraic software package such as *DERIVE*, which we used).

2. Circle BPQ and the point U

We take ABC as triangle of reference and as we are first considering any quadrilateral we take D to have co-ordinates D(f, g, h). We also take P on AB to have co-ordinates (p, 1 – p, 0) and Q on BC to have co-ordinates Q(0, q, 1 – q). Eventually we shall suppose D lies on circle ABC and then we set $f = -a^2t(1-t)$, $g = b^2(1-t)$, $h = c^2t$, where t is a parameter, $-\infty < t <= \infty$.

The circle BPQ has an equation of the form

$$a^2yz + b^2zx + c^2xy + (lx + my + nz)(x + y + z) = 0, \qquad (2.1)$$

where we find $l = -c^2(1-p)$, $m = 0$, $n = -a^2q$. (2.2)

The diagonal BD, with equation hx = fz, meets circle BPQ at the point U with co-ordinates (x, y, z), where

$$x = f(c^2fp + a^2h(1-q)),\ y = a^2hq(f + h) - f(b^2h - c^2(f + h)(1-p)),\ z = h(c^2fp + a^2h(1-q)). \qquad (2.3)$$

3. Circle SDU and the points R and M

The equation of AD is hy = gz, so we may take S to have co-ordinates S(s, g, h). Circle SDU now has an equation of the form (2.1), where we find

$$l = (a^2gh - (g + h)(b^2h + c^2g))/((f + g + h)(g + h + s)), \qquad (3.1)$$

$$m = (a^2h(q-1) - c^2fp)/(f + g + h), \qquad (3.2)$$

$$n = -(a^2gh(f + q(g + h + s) + b^2fhs - c^2fg(gp + hp + s(p - 1))/(h(f + g + h)(g + h + s)). \qquad (3.3)$$

The equation of CD is gx = fy and this meets circle SDU at the point R with co-ordinates (x, y, z), where

$$x = f(a^2gh(f + q(g + h + s)) + f(b^2hs - c^2g(gp + hp + s(p - 1)))), \qquad (3.4)$$

$$y = g(a^2gh(f + q(g + h + s)) + f(b^2hs - c^2g(gp + hp + s(p - 1)))), \qquad (3.5)$$

$$z = f(b^2h(f + g)(g + h) + c^2g(f(gp + hp + s(p - 1)) + g^2p + g(h + s)(p - 1) - h(h + s))) - a^2gh(f^2 + f(gq + (h + s)(q - 1)) + g(g + h + s)(q - 1)). \qquad (3.6)$$

The equation of the common chord of two circles with equations of the form (2.1) with l_1, m_1, n_1 and l_2, m_2, n_2 respectively is

$$(l_1 - l_2)x + (m_1 - m_2)y + (n_1 - n_2)z = 0. \qquad (3.7)$$

Circles BPQ and SDU meet at U and a second point M and from Equations (2.2) and (3.1) – (3.3) we find the equation of the line UM is

$$h(a^2gh - b^2h(g + h) - c^2(f(g + h + s)(p - 1) + (g + h)(gp + (h + s)(p - 1))))x + h(g + h + s)(a^2h(q - 1) - c^2fp)y + a^2h(f(g(q - 1) + q(h + s) + hq(g + h + s)) - f(b^2hs - c^2g(gp + hp + s(p - 1)))z = 0. \qquad (3.8)$$

This line meets circle BPQ at U (whose co-ordinates are given by Equations (2.3) and the point M with co-ordinates (x, y, z), where

$$x = a^2h(g + h + s)(a^2h(q - 1)(g(q - 1) + q(h + s)) + b^2hs(1 - q) + c^2(g^2p(q - 1) + g(hp(2q - 1) + s(p - 1)(q - 1)) + hpq(h + s))), \qquad (3.9)$$

$$y = -a^4gh^2(g(q - 1) + q(h + s)) + a^2h(b^2h(g^2(q - 1) + g(h(2q - 1) + s(q + 1)) + hq(h + s)) + c^2g(g^2p(q - 2) + g(h(p(2q - 3) + 1) + s(2p(q - 1) - q + 2)) + h^2(p(q - 1) + 1) + hs(p(2q - 1) - q + 1) + qs^2(p - 1))) - (g + h)(b^4h^2s - b^2c^2h(g^2p + g(hp - s) + s(1 - p)(h + s)) - c^4g(g^2p^2 + gp(h(2p - 1) + 2s(p - 1)) + (p - 1)(h^2p + hs(2p - 1) + s^2(p - 1)))), \qquad (3.10)$$

$$z = c^2h(g + h + s)(a^2h(g(p(q - 2) - q + 1) + h(p - 1)(q - 1) + s(p - 1)(q - 1)) + p(g + h)(b^2h + c^2(gp + h(p - 1) + s(p - 1)))). \qquad (3.11)$$

4. Circles CRQ and APS pass through M and the points T and I

Circle CRQ has the form of Equation (2.1) and we find

$$l = (a^2gh - b^2h(g + h) - c^2g(gp + hp + s(p - 1)))/(h(g + h + s)), \qquad (4.1)$$

$$m = a^2(q - 1), \qquad (4.2)$$

$$n = 0. \qquad (4.3)$$

It may now be checked that circle CRQ passes through M.

Circle APS has the form of Equation (2.1) and we find

$$l = 0, \quad (4.4)$$
$$m = – c^2p, \quad (4.5)$$
$$n = – (a^2gh + b^2hs – c^2g(gp + hp + s(p – 1)))/(h(g + h + s)). \quad (4.6)$$

It may now be checked that circle APS passes through M. Hence M is a common point of all four circles.

The circle APS meets AC with equation y = 0 at A and the point T with co-ordinates (x, y, z), where

$$x = – a^2gh – b^2hs + c^2g(gp + hp + s(p – 1)), \quad (4.7)$$
$$y = 0, \quad (4.8)$$
$$z = a^2gh – b^2h(g + h) – c^2g(gp + hp + s(p – 1)). \quad (4.9)$$

It may now be checked that circle CRQ passes through T.

The diagonals AC and BD meet at I(f, 0, h).

It has now been established that if ABCD is any quadrilateral and arbitrary points P, Q, S are chosen on AB. BC, DA then points R, T, U exist on CD, CA, BD respectively and a point M exists so that circle BPQ passes through M and U, circle DSU passes through R and M, circle CRQ passes through T and M and circle APS passes through T and M. Thus M is a common Miquel point of the four triangles ABD, BCD, ACD and ABC.

5. The centres E, F, G, H of the circles APS, BPQ, CRQ, DRS

The centre of a circle whose equation is of the form (2.1) has co-ordinates (x, y, z) given by the equations

$$x = – (a^4 – a^2(b^2 + c^2 + 2l – m – n) + (b^2 – c^2)(m – n)), \quad (5.1)$$
$$y = – (b^4 – b^2(c^2 + a^2 + 2m – n – l) + (c^2 – a^2)(n – l)), \quad (5.2)$$
$$z = – (c^4 – c^2(a^2 + b^2 + 2n – l – m) + (a^2 – b^2)(l – m)). \quad (5.3)$$

Using the values of l, m, n given by Equations (2.2) we find the co-ordinates of F, the centre of circle BPQ to be (x, y, z), where

$$x = a^2(a^2(q – 1) + b^2(1 – q) + c^2(2p + q – 1)), \quad (5.4)$$
$$y = – a^4q + a^2(b^2(q + 1) – c^2(p – q – 1)) – b^4 + b^2c^2(2 – p) + c^4(p – 1), \quad (5.5)$$
$$z = c^2(p(b^2 – c^2) – a^2(p + 2q – 2)). \quad (5.6)$$

Using the values of l, m, n given by Equations (3.1) – (3.3) we find the co-ordinates of H, the centre of circle DRS to be (x, y, z), where

$$x = -(1/(h(f + g + h)(g + h + s)))(a^4h(f(h + s) + g^2(1 - q) - g(h + s(q - 1)) + hq(h + s))$$
$$- a^2(b^2h(f(h + 2s) + g^2(1 - q) - g(h(2q - 1) + s(q - 1)) - h(hq + s(q - 2)))$$
$$+ c^2(h(g^2(q - 1) + g(h(2q - 1) + s(q + 1)) + hq(h + s)) - f(g^2p + g(s(p - 1) - 2h)$$
$$- h(h + s)(p + 1)))) + f(b^2 - c^2)(b^2hs - c^2(g^2p + g(2hp + s(p - 1)) + hp(h + s)))), \quad (5.7)$$

$$y = -(1/(h(f + g + h)(g + h + s)))(a^4gh(f + gq + h(q + 1) + qs) - a^2(b^2h(f(2g + h) + g^2(q + 1)$$
$$+ g(3hq + s(q + 1)) + h(2hq + s(2q - 1))) + c^2g(f(gp + h(p + 1) + s(p - 1)) + h(g(q + 1)$$
$$+ h(q + 2) + qs))) + b^4h(f + g + s)(g + h) + b^2c^2(f(g^2p + g(h(3p - 1) + s(p - 1))$$
$$+ h(h(2p - 1) + 2s(p - 1))) - h(g + h)(2g + s)) + c^4g(f(gp + hp + s(p - 1)) + h(g + h))).$$
$$(5.8)$$

$$z = (1/(h(f + g + h)(g + h + s))(a^4h^2(g(q - 2) + (h + s)(q - 1)) - a^2h(b^2h(g(q - 3) + h(q - 2)$$
$$+ s(q - 1)) + c^2(f(g(p + 1) + (h + s)(p - 1)) + 2g^2(q - 1) + g(3h(q - 1) + s(2q - 1))$$
$$+ h(h + s)(q - 2))) - b^4h^2(g + h) + b^2c^2h(f(g(p + 1) + h(p + 1) + s(p - 1)) + (g + h)(2h + s))$$
$$+ c^4(f(2g^2p + g(h(3p - 1) + 2s(p - 1)) + h(h + s)(p - 1)) - h(g + h)(h + s))). \quad (5.9)$$

Using values of l, m, n given by Equations (4.1) – (4.3) we find the co-ordinates of G, the centre of circle CRQ are (x, y, z), where

$$x = -(1/(h(g + h + s)))(a^2(a^2h(g(q - 2) + q(h + s)) + b^2h(gq + hq + s(q - 2))$$
$$+ c^2(2g^2p + g(h(2p - q) + 2s(p - 1)) - hq(h + s)))), \quad (5.10)$$

$$y = -(1/(h(g + h + s)))(a^4gh - a^2(b^2h(g(2q - 1) + 2hq + s(2q - 1)) + c^2g(gp + h(p + 1) + s(p - 1)))$$
$$+ b^4hs - b^2c^2(g^2p + g(hp + s(p - 1)) + hs) + c^4g(gp + hp + s(p - 1))), \quad (5.11)$$

$$z = (1/(h(g + h + s)))(a^4h(g(q - 2) + (h + s)(q - 1)) + a^2(c^2(g^2p + g(h(p - q + 1) + s(p - 1))$$
$$+ h(2 - q)(h + s)) - b^2h(g(q - 3) + h(q - 2) + s(q - 1))) - b^4h(g + h)$$
$$- b^2c^2(g^2p + g(h(p - 2) + s(p - 1)) - h(2h + s)) + c^4(g^2p + g(h + s)(p - 1) - h(h + s))). \quad (5.12)$$

Using values of l, m, n given by Equations (4.4) – (4.6) we find the co-ordinates of E, the centre of circle APS are (x, y, z), where

$$x = -(1/(h(g + h + s)))(a^4h(h + s) + a^2(c^2(g^2p + g(s(p - 1) - 2h) - h(h + s)(p + 1)) - b^2h(h + 2s))$$
$$+ (b^2 - c^2)(b^2hs - c^2(g^2p + g(2hp + s(p - 1)) + hp(h + s)))), \quad (5.13)$$

$$y = -(1/(h(g + h + s)))(a^4gh - a^2(b^2h(2g + h) + c^2g(gp + h(p + 1) + s(p - 1))) + b^4h(g + h)$$
$$+ b^2c^2(g^2p + g(h(3p - 1) + s(p - 1)) + h(h(2p - 1) + 2s(p - 1))) + c^4g(gp + hp + s(p - 1)))$$
$$(5.14)$$

$$z = -(1/(h(g + h + s)))(c^2(a^2h(g(p + 1) + (h + s)(p - 1)) - b^2h(g(p + 1) + h(p + 1) + s(p - 1))$$
$$- c^2(2g^2p + g(h(3p - 1) + 2s(p - 1)) + h(h + s)(p - 1)))). \quad (5.15)$$

Using

$$f = -a^2t(1 - t),\ g = b^2(1 - t),\ h = c^2t, \quad (5.16)$$

so that D lies on circle ABC it may now be shown that E, F, G, H are concyclic. In general they are not concyclic. The working is technically very difficult and lengthy (covering about 22 pages of detailed working in *DERIVE*, which cannot be relied upon to be typed accurately – an unfortunate property of *DERIVE* is that one cannot copy from it into a Word text).

6. The points I, T, M, U are concyclic when ABCD is a cyclic quadrilateral

The co-ordinates of U are given by Equations (2.3) and using Equations (5.16) these become

$$x = a^4c^2t^2(t-1)(p(t-1)-q+1) \quad (6.1)$$

$$y = -a^2c^2t^2(a^2(t-1)(p(t-1)-q-t+1)+b^2(t-1)+c^2(p(t-1)-q-t+1)), \quad (6.2)$$

$$z = a^2c^4t^2(p(t-1)-q+1). \quad (6.3)$$

The co-ordinates of M are given by Equations (3.9) – (3.11) and using Equations (5.16) these become

$$x = a^2c^4t(a^2t(q-1)(b^2(q-1)(t-1)-q(c^2t+s))+b^4p(1-q)(t-1)^2+b^2(c^2pt(t-1)(2q-1)$$
$$+s(q-1)(p(t-1)+1))-c^2pqt(c^2t+s))(b^2(t-1)-c^2t-s), \quad (6.4)$$

$$y = a^4b^2c^4t^2(1-t)(b^2(q-1)(t-1)-q(c^2t+s))-a^2b^2c^4t(b^4(t-1)^2(p(q-2)(t-1)-t(q-1))$$
$$+b^2(1-t)(c^2t(p(t-1)(2q-3)-2qt+2t-1)+s(2p(q-1)(t-1)+q(1-2t)+t-2))$$
$$+c^4t^2(p(q-1)(t-1)-qt+t-1)+c^2st(p(t-1)(2q-1)+q(1-2t)+t-1)$$
$$+qs^2(p-1)(t-1))+b^2c^4(b^2(t-1)-c^2t)(b^4p(t-1)^2(p(t-1)-t)$$
$$-b^2(c^2pt(t-1)(2p(t-1)-2t+1)+s(2p^2(t-1)^2-2p(t-1)^2-t))+(p-1)(c^4pt^2(t-1)$$
$$+c^2st(2p(t-1)+1)+s^2(p(t-1)+1))), \quad (6.5)$$

$$z = c^6t(a^2t(b^2(t-1)(p(q-2)-q+1)+c^2t(1-q)(p-1)+s(1-q)(p-1))$$
$$+p(b^2(p(t-1)-t)+(1-p)(c^2t+s))(c^2t-b^2(t-1)))(b^2(t-1)-c^2t-s). \quad (6.6)$$

The co-ordinates of I given in Section 4 become

$$x = -a^2t(1-t),\ y = 0,\ z = c^2t. \quad (6.7)$$

The co-ordinates of T given by Equations (4.7) – (4.9) become on using Equation (5.16)

$$x = a^2b^2c^2t(t-1)+b^2c^2(b^2p(t-1)^2+c^2pt(1-t)-s(p(t-1)+1)), \quad (6.8)$$

$$y = 0, \quad (6.9)$$

$$z = a^2b^2c^2t(1-t)-b^2c^2(b^2(t-1)(p(t-1)-t)+c^2t(t-p(t-1))+s(1-t)(p-1)). \quad (6.10)$$

It may now be shown that U, M, I, T lie on a circle with equation of the form (2.1) with

$$l = b^2c^2(a^2t(t-1)+b^2(t-1)(p(t-1)-t)+c^2t(t-p(t-1))+s(1-t)(p-1))$$

all divided by $t(a^2(t-1)+c^2)(b^2(t-1)-c^2t-s)$, (6.11)

$$m = -(a^2c^2(p(t-1)-q+1))/(a^2(t-1)+c^2), \quad (6.12)$$

$$n = a^2b^2(1-t)(a^2t(t-1)+b^2p(t-1)^2+c^2pt(1-t)+ps(1-t)-s),$$

all divided by $t(a^2(t-1)+c^2)(b^2(t-1)-c^2t-s)$. (6.13)

The points I, T, M, U are concyclic whether or not ABCD is a circle. David Monk has kindly pointed out that EG is the line of centres of circles

APS and CQR and therefore bisects the common chord MT at right angles. Similarly FH bisects MU at right angles and hence the chords EG, FH intersect at K, the centre of circle ITMU.

Fig. 2 shows the construction when the quadrilateral is re-entrant.

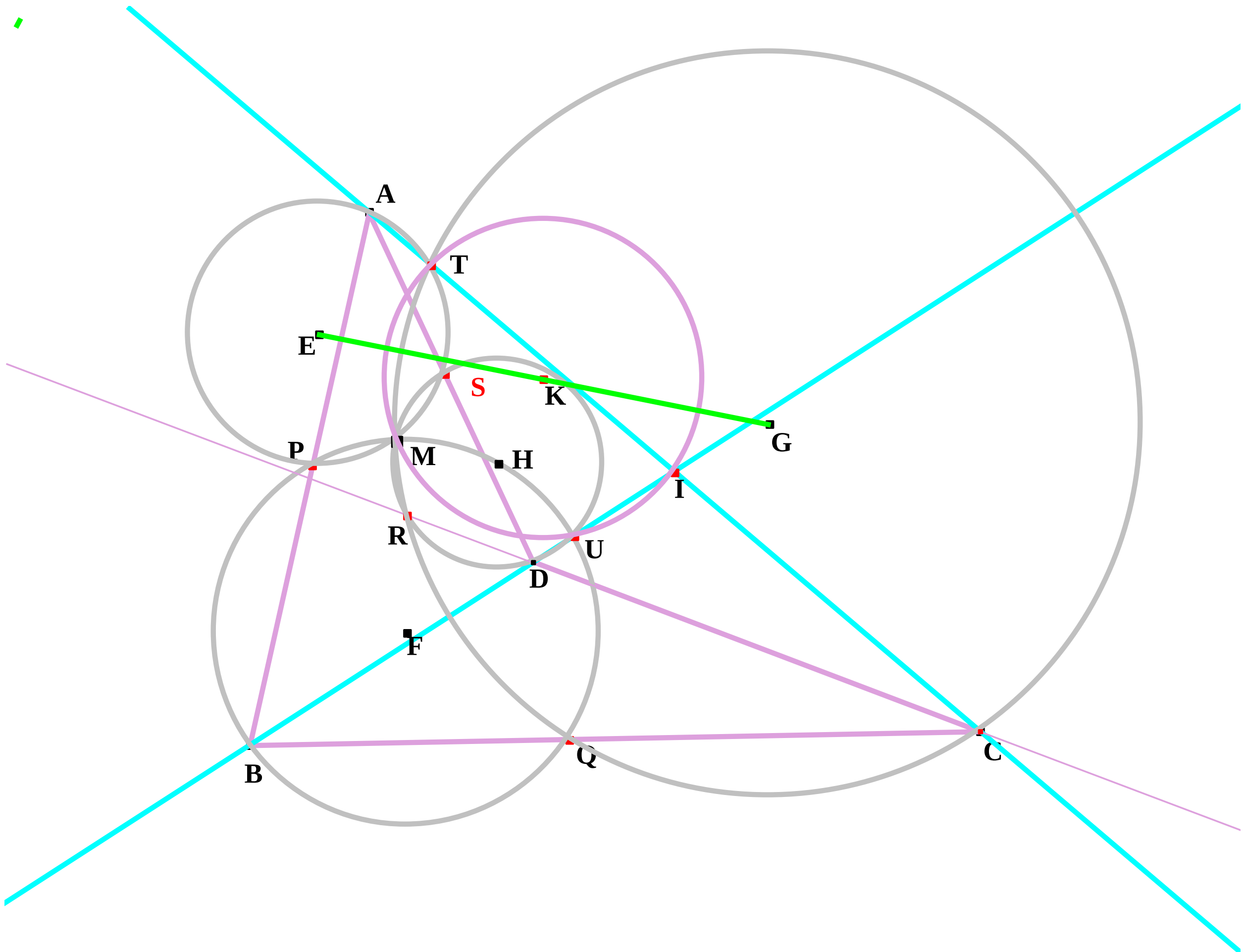

Fig. 2

Showing the common point M when ABCD is re-entrant

Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL, BS8 2SP.