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Four Nine-Point Circles

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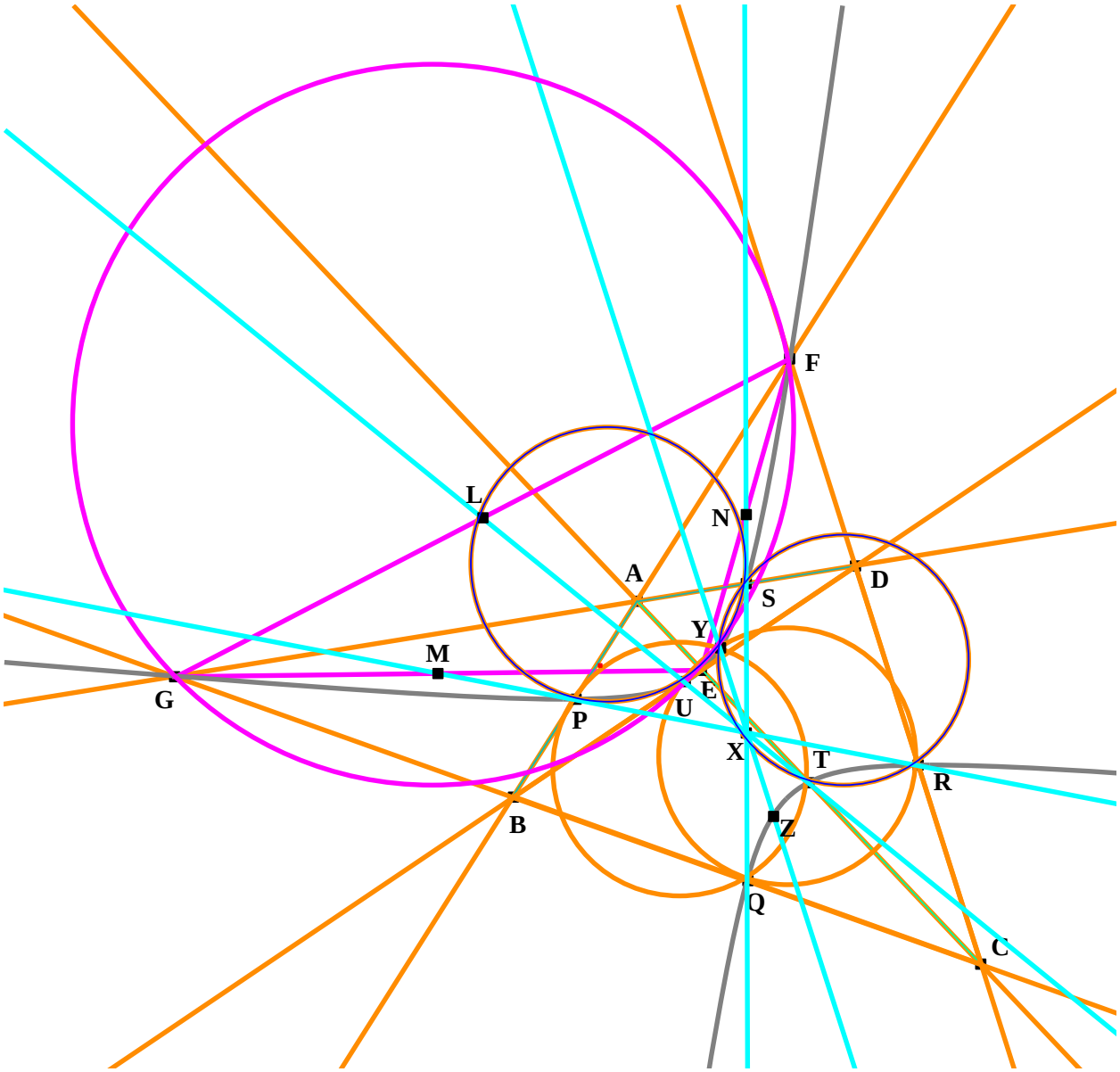

Fig.1

Nine-Point Circles intersect at a point on the circumcircle of the Diagonal Point Triangle

Abstract: In any quadrilateral ABCD the four nine-point circles of triangles BCD, ACD, ABD, ABC have a point in common that lies on the hyperbola through the mid-points and on the circumcircle of the diagonal point triangle.

1. Introduction

Given a quadrilateral ABCD (with neither pair of opposite sides parallel) in which P, Q, R, S, T, U are the midpoints of sides AB, BC, CD, DA, AC , BD and diagonal point triangle EFG, then a hyperbola Σ passes through the nine points P, Q, R, S, T, U, E, F, G. We also prove that the nine-point circles of triangles BCD, ACD, ABD, ABC have a common point which lies on Σ and also on the circumcircle of triangle EFG. Areal co-ordinates are used with ABC as triangle of reference. Note that the quadrilateral does not have to be convex and when, for example, D lies in triangle ABC then the diagonal point triangle is the triangle formed by feet of the Cevians of D in triangle ABC.

2. Co-ordinates of points

With ABC as triangle of reference we have A(1, 0, 0, B(0, 1, 0), C(0, 0, 1). Let D have co-ordinates D(f, g, h) with f + g + h = 1. Unnormalised co-ordinates for other points are soon determined and are: P(1, 1, 0), Q(0, 1, 1), R(f, g, 1 + h), S(1 + f, g, h), T(1, 0, 1), U(f, 1 + g, h), E(f, 0, h), F(f, g, 0), G(0, g, h).

3. Conic Σ through the midpoints and E, F, G and its centre X

The conic Σ through P, Q, R, S, T, U, E, F, G is of the form

$$ux^2 + vy^2 + wz^2 + 2lyz + 2mzx + 2nxy = 0, \qquad (3.1)$$

where it is easily checked that

$$u = 2gh,\ v = 2hf,\ w = 2fg,\ l = -f(g + h),\ m = -g(h + f),\ n = -h(f + g)). \qquad (3.2)$$

Using the formula $x = vw - mv - nw - l^2 + lm + nl$ for the centre X we obtain $x = 2f + g + h$. Similarly $y = f + 2g + h$ and $z = f + g + 2h$.

4. The circumcircle of triangle EFG

This has equation

$$\begin{aligned} &gh(a^2gh(f + g)(h + f) - f(g + h)(b^2h(f + g) + c^2g(f + h)))x^2 + \ldots + \ldots \\ &- (g + h)(a^2gh(f + g)(h + f) + f(h - g)(b^2h(f + g) - c^2g(h + f)))yz - \ \ldots\ - \ \ldots = 0. \qquad (4.1) \end{aligned}$$

where the dots imply additional terms to be obtained by cyclic change of x, y, z and f, g, h and a, b, c.

5. The nine-point circle of triangle ABC

This has equation

$$(b^2 + c^2 - a^2)x^2 + (c^2 + a^2 - b^2)y^2 + (a^2 + b^2 - c^2z^2 - 2a^2yz - 2b^2zx - 2c^2xy = 0. \qquad (5.1)$$

This meets Σ at the point Y with co-ordinates (x, y, z), where

$$x = f(a^2(g - h) - (b^2 - c^2)(g + h))(c^2g(h + f) - b^2h(f + g)), \qquad (5.2)$$

and y, z may be obtained from Equation (5.2) by cyclic change of a, b, c and f, g, h.

It may now be checked that Y lies on the circumcircle of triangle EFG. Similarly Y lies on the nine-point circles of triangles ABD, ACD and BCD.

6. L, M, N the midpoints of the sides FG, GE, EF

L is the midpoint of FG and therefore has co-ordinates (f(g + h), g(f + 2g + h), h(f + g)). The points M, N have co-ordinates that may be written down from those of L by cyclic change of x, y, z and f, g, h. It may now be checked that L, U, X, T are collinear with UX = XT. Similarly M, P, X, R are collinear with PX = XR and N, S, X, Q are collinear with SX = XQ.

The point Z on YX such that YX = XZ also lies on the hyperbola Σ (since X is the centre) and if ABCD is cyclic Z becomes the circumcentre of ABCD.

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