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Three Centroids created by a Cyclic Quadrilateral

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Abstract: The centroid of the quadrilateral considered to be an area of constant density is G, the centroid of the quadrilateral considered as having unit masses at its vertices is N, the centroid of the quadrilateral considered as having unit masses at its vertices and mass of two units at E (the intersection of its diagonals) is F. It is shown that E, F, N, G are collinear.

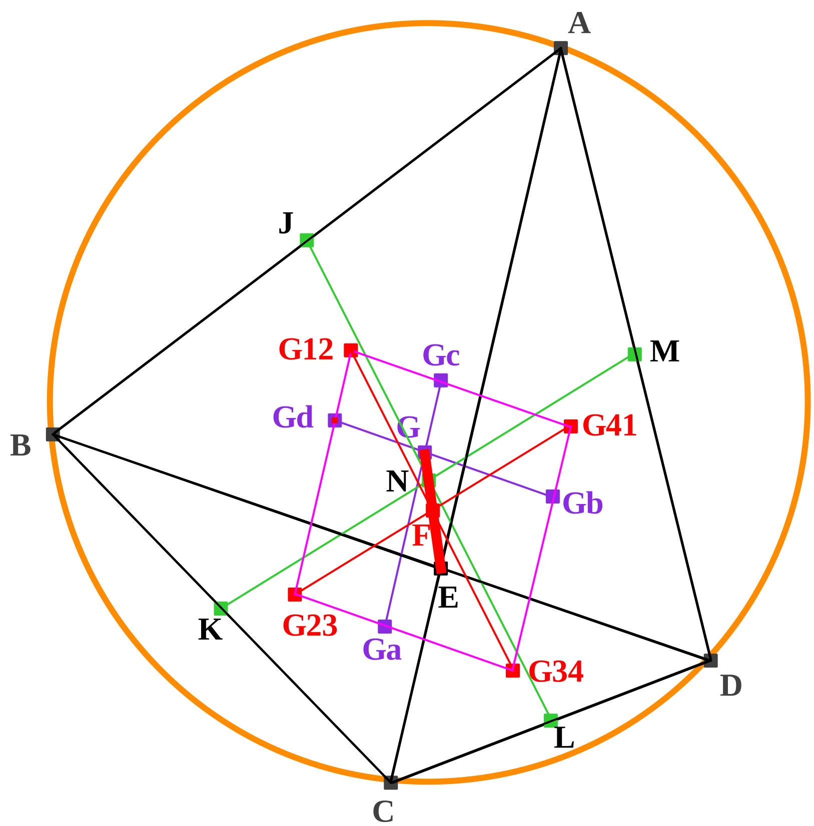

Fig. 1
Centroids in a Cyclic Quadrilateral

1. Introduction

In the diagram above ABCD is a cyclic quadrilateral in which the diagonals meet at the point E. Points G_a, G_b, G_c, G_d are the centroids of triangles BCD, ACD, ABD, ABC respectively. G_aG_c meets G_bG_d at the point G. Points J, K, L, M are the midpoints of AB, BC, CD, DA respectively.

JL meets KM at the point N. Points G_{12}, G_{23}, G_{34}, G_{41} are the centroids of triangles ABE, BCE, CDE, DAE respectively. $G_{12}G_{34}$ meets $G_{23}G_{41}$ at F. The centroid of the quadrilateral considered to be an area of constant density is G, the centroid of the quadrilateral considered as having unit masses at its vertices is N, the centroid of the quadrilateral considered as having unit masses at its vertices and a mass of two units at E (the intersection of its diagonals) is F. It is shown that E, F, N, G are collinear. We use oblique axes with x-axis in the direction EA and with y-axis in direction EB. Thus E has co-ordinates (0. 0) and A, B, C, D may be chosen to have co-ordinates (fg, 0), (0, hf), (– hk, 0), (0, – gk), where f, g, h, k are any constants and the co-ordinates reflect the relation AE x EC = BE x ED implying, by the intersecting chord theorem, that ABCD is cyclic.

2. The centroid G of ABCD, considered to be an area of uniform density

The centroid G_d of triangle ABC has co-ordinates (1/3)(fg – hk, fh).
The centroid G_b of triangle ACD has co-ordinates (1/3)(fg – hk, – gk).
The centroid G_c of triangle ABD has co-ordinates (1/3)(fg, fh – gk).
The centroid G_a of triangle BCD has co-ordinates (1/3)(– hk, fh – gk).

Since ABC and ACD combine to give ABCD it follows that the centroid of ABCD lies on G_dG_b. Similarly it lies on G_cG_a. It follows immediately, by inspection, that $G = G_dG_b{}^\wedge G_cG_a$ has co-ordinates (1/3)(fg – hk, fh – gk). The line EG has equation

$$(fh - gk)x = (fg - hk)y. \tag{2.1}$$

3. The centroid F when there is a mass 1 at each of A, B, C, D and a mass 2 at E

These masses are arranged so that we may suppose there are equal masses at the vertices of each of the triangles ABE, BCE, CDE, DAE. This means that if the centroids of these triangles are G_{12}, G_{23}, G_{34}, G_{41} respectively then F will be the intersection of the lines $G_{23}G_{41}$ and $G_{12}G_{34}$.

The centroid G_{12} of ABE has co-ordinates (1/3)(fg, fh).
The centroid G_{23} of BCE has co-ordinates (1/3)(– hk, fh).
The centroid G_{34} of CDE has co-ordinates (1/3)(– hk, – gk).
The centroid G_{41} of DAE has co-ordinates (1/3)(fg, – gk).

The line $G_{23}G_{41}$ has equation

$$3(fh + gk)x + 3(fg + hk)y = gh(f^2 - k^2). \tag{3.1}$$

The line $G_{12}G_{34}$ has equation

$$3(fh + gk)x - 3(fg + hk)y = fk(g^2 - h^2). \tag{3.2}$$

The co-ordinates of $F = G_{23}G_{41}\wedge G_{12}G_{34}$ are (1/6)(fg – hk, fh – gk). It is immediately obvious that F lies on EG and EG = 2EF.

4. The centroid N when there are equal masses at each of A, B, C, D

N has co-ordinates a quarter of the sum of those of A, B, C, D and are therefore (1/4)(fg – hk, fh – gk). If J, K, L, M are the midpoints of AB, BC, CD, DA respectively then N is also the intersection of JL and KM. Evidently N lies on EG and is such that EN = 1.5 x EF.

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