Article 14

Intersecting Circles having chords the sides of a Cyclic Quadrilateral

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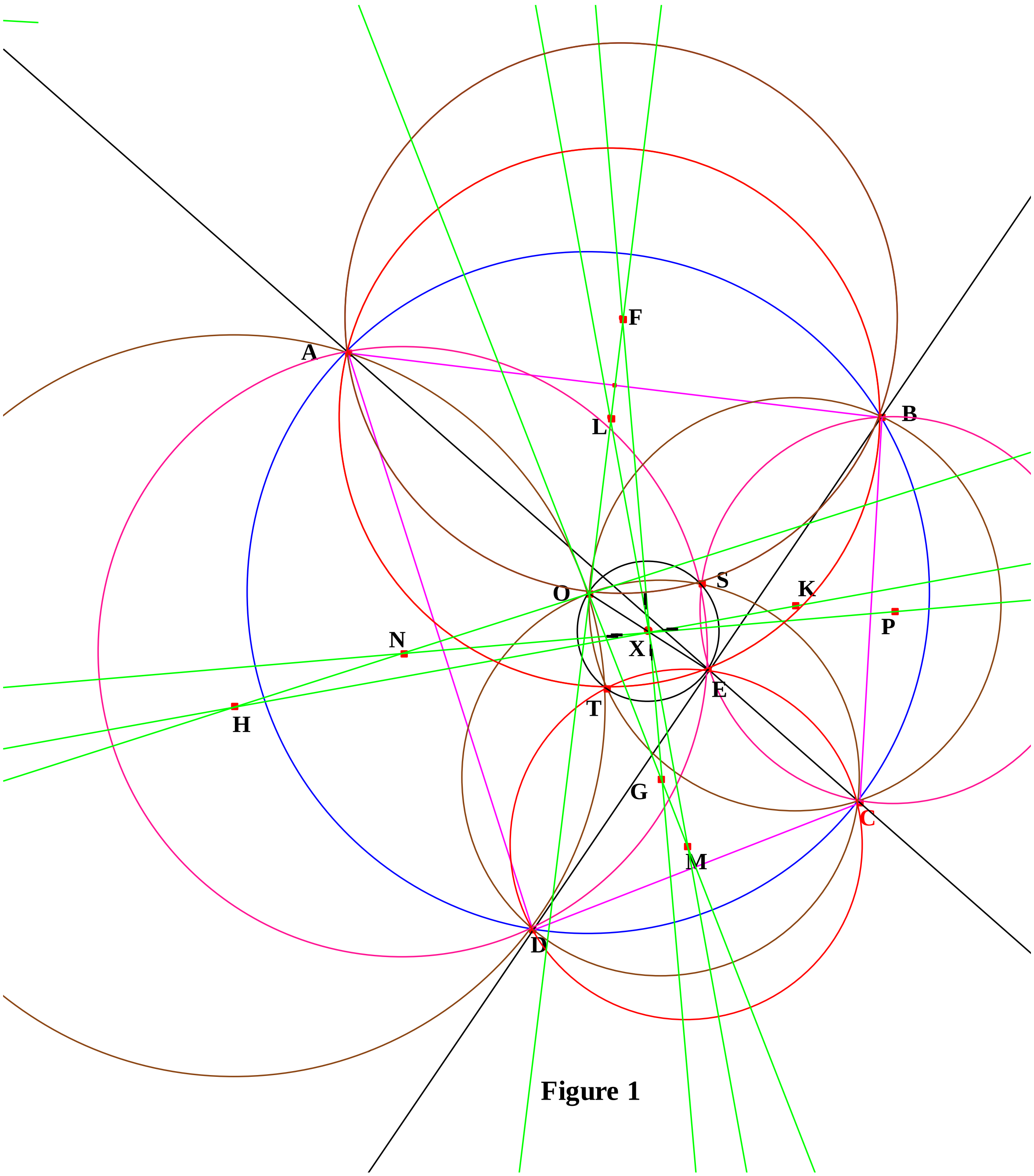

Figure 1

1. Introduction

We consider a cyclic quadrilateral in which the centre O of the circle ABCD is distinct from the intersection E of the diagonals AC and BD.

In this configuration a number of results hold.

(i) Circles AOB, COD, AED, BEC are concurrent at a point S;
(ii) Circles AOD, BOC, AEB, CED are concurrent at a point T;
(iii) O, S, E, T are concyclic;
(iv) The centre of circle OSET is X, the midpoint of OE, and it follows that angle OSE and OTE are each 90^o;
(v) FXG, NXP, HXK, LXM are straight lines, where F, G, H, K, L, M, N, P are the centres of circles AOB, COD, AOD, BOC, AEB, CED, AED, BEC respectively;
(vi) The following angles are each 90^o: angles HXL, NXF, KXL, PXF, HXM, NXG, KXM and PXG and consequently angles HXN, FXL, GXM, KXP are equal.

See Figure 1 for an illustration of these properties.

Now let AB and CD meet at F, and let AD and BC meet at G. Then another set of properties hold. In properties (vii) to (xi) points S, T, E are as before but points F, G, H, K are different points.

(vii) Circles FBC, FAD, GCD, GAB meet at a point H lying on FG;
(viii) Circles FBD, GAC pass through S and circles FAC, GBD pass through T;
(ix) Points O, E, H are collinear;
(x) OS^ET = F and OT^SE = G;
(xi) Points F, G, S, T are concyclic and the centre K of the circle FGST is the midpoint of FG.

See Figure 2 for an illustration of these properties.

In the following sections we prove these results, using Cartesian co-ordinates.

2. Circles AOB and COD and the point S

We take the equation of ABCD to be $x^2 + y^2 = 1$ and the co-ordinates of A to be $(2a/(1 + a^2), (1 - a^2)/(1 + a^2))$, with B, C, D having similar co-ordinates with parameters b, c, d respectively. The co-ordinates of O are, of course, (0, 0).

Since it passes through O the equation of circle OAB is of the form

$$x^2 + y^2 + 2gx + 2fy = 0. \qquad (2.1)$$

Inserting the co-ordinates of A and B in Equation (2.1) we find f and g and substituting back we obtain the equation of circle OAB to be

$$(1 + ab)(x^2 + y^2) - (a + b)x - (1 - ab)y = 0. \qquad (2.2)$$

The equation of OCD may be obtained from Equation (2.2) by writing c, d instead of a, b. The intersections of these two circles are the origin O and the point labelled S, whose co-ordinates are (x, y), where

$$x = \left(\frac{1}{k}\right)\left(2(ab - cd)(a(b(c+d) - cd + 1) + b(1-cd) - c - d)\right), \tag{2.3}$$

$$y = -\left(\frac{1}{k}\right)(a^2(b^2(c^2 + 2cd + d^2) - 2bcd(c+d) + c^2d^2 - 1) - 2a(b^2cd\,(c+d) + b(1-c^2d^2) - c - d) + b^2(c^2d^2 - 1) + 2b(c+d) - (c+d)^2) \tag{2.4}$$

and

$$k = a^2(b^2(c^2 + 2cd + d^2 + 4) - 2b(c+d)(1+cd) + (1+cd)^2) - 2a(b^2(c^2d + c(d^2+1) + d) - b(1+c^2)(1+d^2) + c^2d + c(d^2 + 1) + d) + b^2(c^2d^2 + 2cd + 1) - 2b(c^2d + c(d^2 + 1) + d) + c^2(4d^2 + 1) + 2cd + d^2 \tag{2.5}$$

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3. Circles AOD and BOC and the point T

The equations of circles AOD and BOC may be written down by altering the parameters in Equation (2.2) to a, d and to b, c respectively. The co-ordinates of T may be written down from Equations (2.3) to (2.5) by exchanging a and c (or b and d, but not both).

4. The point of intersection E of the diagonals AC and BD

The equation of AC is

$$x(a+c) + y(1-ac) = 1 + ac \tag{4.1}$$

and the equation of BD is

$$x(b+d) + y(1-bd) = 1 + bd. \tag{4.2}$$

These lines meet at E with co-ordinates (x, y), where

$$x = \left(\frac{1}{k}\right)2(ac - bd), \tag{4.3}$$

$$y = -\left(\frac{1}{k}\right)(a(b(c-d) + cd - 1) + b(1-cd) - c + d) \tag{4.4}$$

and

$$k = a(b(c-d) + cd + 1) - b(cd+1) + c - d. \tag{4.5}$$

5. The circles AED, BEC, AEB, CED

The general equation of a circle is of the form

$$x^2 + y^2 + 2gx + 2fy + k = 0. \tag{5.1}$$

If we now insert the co-ordinates of A, E, D into Equation (5.1) we obtain three equations for f, g, k. When these are substituted back into Equation (2.1) we find the equation of AED, which is

$$\left(a\left(b\left(c-d\right)+\ cd+1\right)-\ b(cd+1)+\ c-d\right)(x^2+\ y^2)+\ 2x(a+d)(b-c)+$$
$$2y(c-b)(ad-1)-\ a(b(c+d)-\ cd+1)+\ b(cd-1)+\ c+d\ =0$$
. (5.2)

The equations of circles BEC, AEB, CED may now be written down from Equation (5.2). Bearing in mind that the co-ordinates of E are invariant under the interchange of a and c and of b and d, it is now straightforward to write down the equations of circles BEC, AEB, CED.

6. (i) and (ii) and the circle OSET

It may now be verified that circles AED and BEC pass through S and circles AEB and CED pass through T.

The circle OES has an equation of the form (2.1) and inserting the co-ordinates of E, given in Equations (4.3) to (4.5) and the co-ordinates of S, given in Equations (2.3) to (2.5) we get two equations for f and g. When these values are reinserted in (2.1) we obtain the equation of OES, which is

$$(a(b(c-d)+\ cd+1)-\ b(cd+1)+\ c-d)(x^2+\ y^2)+\ 2x(bd-ac)+ y(a(b(c-d)+\ cd-1)+\ b(1-cd)-\ c+d)=0. \tag{6.1}$$

It may now be verified that T lies on this circle.

7. Centres of circles and properties (iv), (v), (vi)

The centre of the circle with Equation (5.1) is, of course (– g, – f). It follows that the centre X of circle OSET has co-ordinates

$$(1/k)(ac-bd, -½(a(b(c-d)+\ cd-1)+\ b(1-cd)-\ c+d)), \tag{7.1}$$

where

$$k=a(b(c-d)+\ cd+1)-\ b(cd+1)+\ c-d. \tag{7.2}$$

Noting Equations (4.3) to (4.5) we see that X is the midpoint of OE. This establishes property (iv).

The perpendicular bisector of AB passes through the centres of circles ABCD, AOB, AEB. It follows that points F, L, O are collinear. Similarly H, N, O and P, K, O and M, G, O are collinear, all four lines meeting at O.

From Equation (2.2) the centre F of circle AOB has co-ordinates (½(a + b)/(1 + ab), ½(1 – ab)/((1 + ab)). Similarly the centre G of circle COD has co-ordinates (½(c + d)/(1 + cd), ½(1 – cd)/((1 + cd)). A 3 x3 determinantal test using the co-ordinates of X, F, G in the first two columns and 1 as the entry in each row of the third column gives a zero result. Hence F, X, G are collinear. Similarly H, X, K are collinear.

From Equation (5.2) the centre N of circle AED has co-ordinates (1/k)((a + d)(c – b), (c – b)(1 – ad)), where k = a(b(c – d) + cd + 1) – b(1 + cd) + c – d. The co-ordinates of the centre P of circle BEC may now be written down by exchanging b and d and also a and c. It may now be verified that N, X, P are collinear and similarly L, X, M are collinear. This now establishes property (v).

We now have the co-ordinates of N, X and F and can therefore find the equations of the lines FX and NX. The equation of FX is

$$4(ab-cd)x-2(a(b(c+d)-cd-1)-b(cd+1)+c+d)y \\ -a(b(c+d)-cd+1)+b(cd-1)+c+d=0, \qquad (7.3)$$

and the equation of NX is

$$(a(b(c+d)-cd-1)-b(cd+1)+c+d)(a(b(c-d)+cd+1)-b(cd+1)+c-d)x+2(ab-cd)(a(b(c-d)+cd+1)-b(cd+1)+c-d)y+(b-c)\big(a^2(b(c-d)-cd-1)+a(b+c)(d^2+1)-d(b(cd+1)+c-d)\big)=0. \qquad (7.4)$$

From these equations it is evident that the product of their gradients is – 1 and hence angle FXN = 90^o. Similarly the angles LXH, PXF, KXL, GXN, MXH, MXK and GXP are all 90^o. This completes the proof of property (vi).

8. The points F and G and the line FG

The intersection of lines AB and CD is the diagonal point F and its co-ordinates are therefore (x, y), where

$$x=\left(\frac{1}{k}\right)2(ab-cd), \qquad (8.1)$$

$$y=-\left(\frac{1}{k}\right)(a(b(c+d)-cd-1)-b(1+cd)+c+d), \qquad (8.2)$$

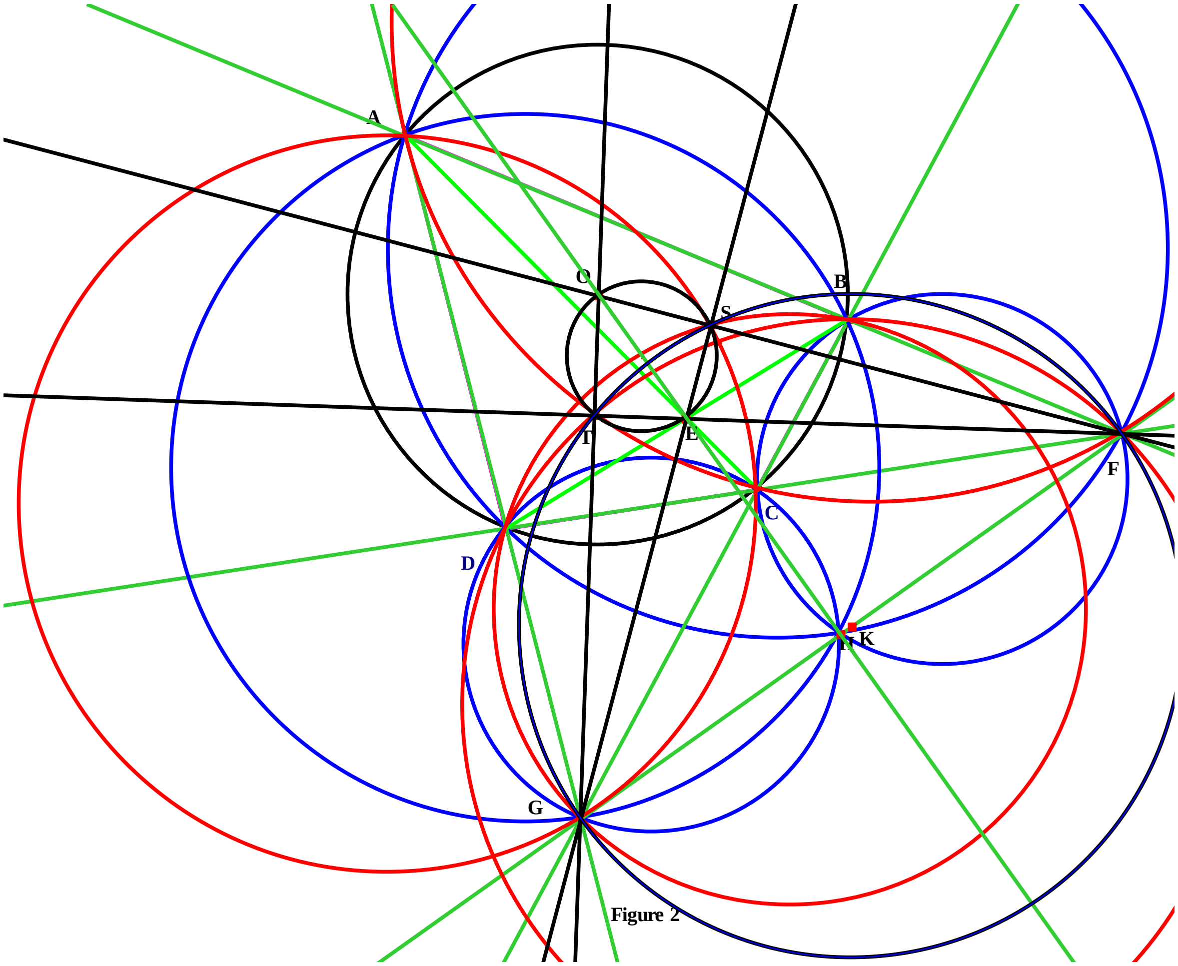

Figure 2

where

$$k = a(b(c+d) - cd + 1) + b(1 - cd) - c - d. \qquad (8.3)$$

The diagonal point G is the intersection of lines AD and BC and therefore has co-ordinates (x, y), where

$$x = \left(\frac{1}{k}\right) 2(bc - ad), \qquad (8.4)$$

$$y = -\left(\frac{1}{k}\right)(a(b(c-d) - cd + 1) + b(cd - 1) - c + d), \qquad (8.5)$$

and

$$k = a(b(c-d) - cd - 1) + b(1 + cd) + c - d. \qquad (8.6)$$

The equation of the line FG may now be obtained and is

$$2(ac - bd)x - (a(b(c-d) + cd - 1) + b(1 - cd) - c + d)y - a(b(c-d) + 1 + cd) + b(1 + cd) - c + d = 0. \qquad (8.7)$$

9. OT and SE pass through G and OS and TE pass through F

The equation of OT is

$$(a(b(c-d)- cd+1)+ b(cd-1)- c+d)x+2(bc-ad)y=0. \qquad (9.1)$$

It may now be verified, using Equations (8.4) to (8 .6) that G lies on this line.

The equation of SE is

$$2(ab-cd)x-(a(b(c+d)- 1-cd)- b(1+cd)+ c+d)y$$
$$-a(b(c+d)- cd+1)+ b(cd-1)+ c+d=0. \qquad (9.2)$$

It may be verified that G also lies on this line. Similarly F lies on OS and TE. This establishes property (x).

10. Circles FBC and FAD, the point H, the line OEH and the circle FSTG

Following the usual procedure we obtain the equation of circle FBC, which is

$$(a(b(c+d)-cd+1)+ b(1-cd)- c-d)(x^2+ y^2)+ 2(d-a)(b+c)x+$$
$$2(d-a)(1-bc)y+a(b(c-d)+ cd+1)- b(1+cd)+ c-d=0.$$

(10.1)

And the equation of circle FAD is

$$(a(b(c+d)-cd+1)+ b(1-cd)- c-d)(x^2+ y^2)+ 2(a+d)(c-b)x+$$
$$2(c-b)(1-ad)y-a(b(c-d)+ cd+1)+ b(1+cd)- c+d=0.$$

(10.2)

The intersection of these circles, other than the point F, is the point we define to be the point H and its co-ordinates are (x, y), where

$$x=\left(\frac{1}{k}\right)2(a^2c(b(c-d)+ cd+1)- a\big(b^2d(c-d)+ b(c^2d+c(d^2+ 1)+ d)-$$
$$c(c-d)\big)+ bd(b(1+cd)- c+d)),$$

(10.3)

$$y= -\left(\frac{1}{k}\right)a^2(b^2(c-d)^2+ 2bcd(c-d)+ c^2d^2- 1)- 2a(b^2cd(c-d)+$$
$$b(c^2d^2- 1)+ c-d)+b^2(c^2d^2- 1)+ 2b(c-d)- (c-d)^2,$$

(10.4)

where

$$k=a^2(b^2(c-d)^2+ 2b(c-d)(cd-1)+ c^2(d^2+ 4)- 2cd+1)- 2a(b^2(c-$$
$$d)(cd-1)+ b(1+c^2)(1+d^2)+ c^2d-c(1+d^2)+ d)+ b^2(c^2d^2- 2cd+4d^2+$$
$$1)+ 2b(c-d)(cd-1)+ (c-d)^2$$

. (10.5)

It may now be verified that H lies on FG with Equation (8.7). The equations of circles GCD and GAB follow by carefully rearranging parameters in Equations (10.1) and (10.2) and then it may be checked that H lies also on these circles. This establishes property (vii).

The line OE passes through H and as O is the orthocentre of triangle EFG it follows that OE cuts FG at right angles.

We have already shown that angles OSE and OTE and that OS passes through F and OT passes through G. It follows that angles FSG and FTG are both right angles. Thus S, T, F, G are concyclic and the centre K of circle FSTG is the midpoint of FG. We have now established properties (ix) and (xi).

11. Circles FBD and FAC pass through T and circles GBD and GAC pass through S

From the co-ordinates of F, B, D we may now obtain the equation of circle FBD, which is

$$(a(b(c+d) - cd + 1) + b(1-cd) - c - d)(x^2 + y^2) + 2(c-a)(b+d)x + 2(c-a)(1-bd)y - a(b(c-d) - cd - 1) - b(1+cd) - c + d = 0 \quad (11.1)$$

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It may now be checked that T lies on this circle. Similarly T lies on circle FAC, S lies on circle GBD and also on circle GAC. This establishes property (viii).

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