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An Extension to the Theory of Hagge Circles

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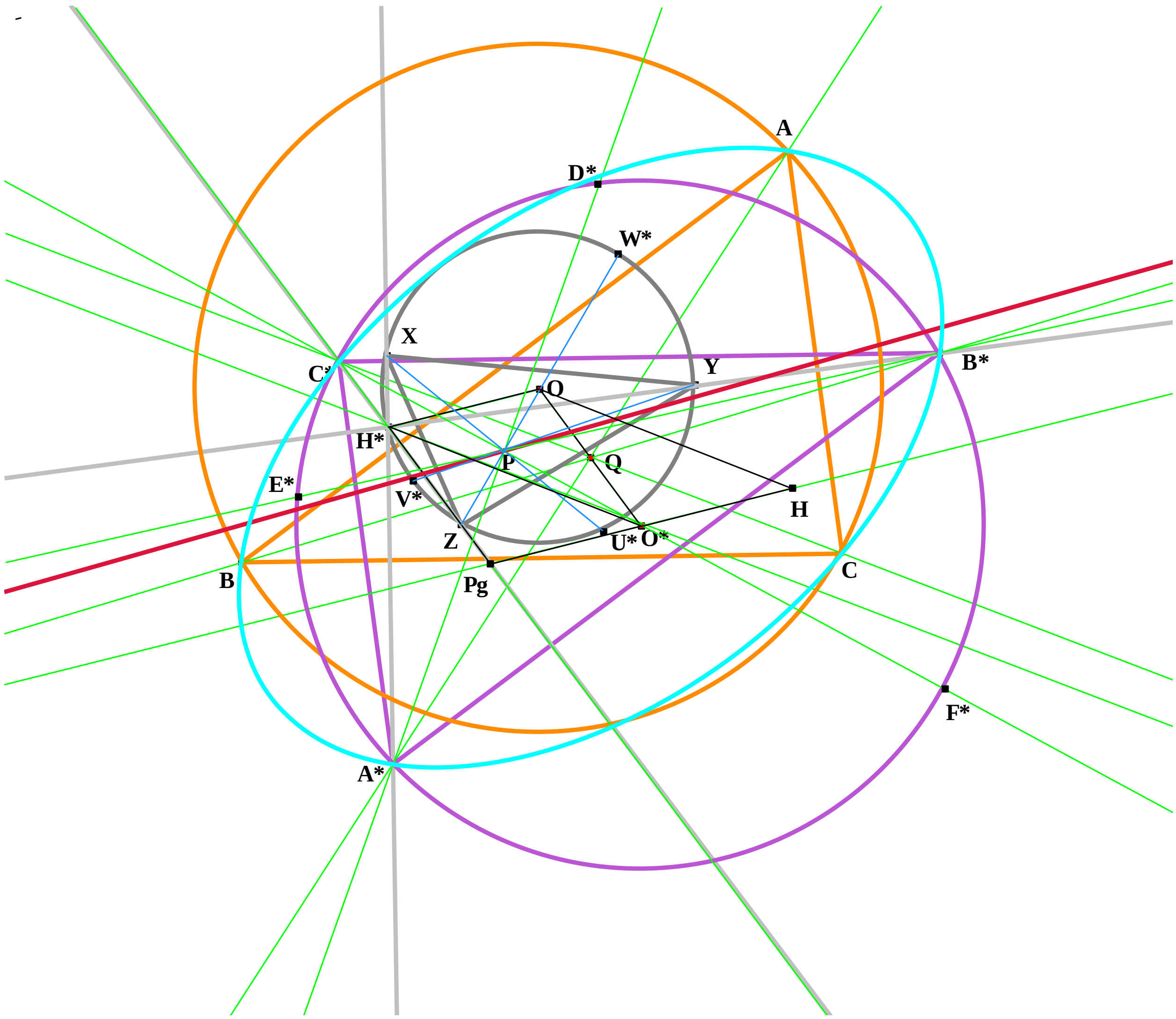

Fig. 1

Two related triangles, a circle with significance in both triangles

1. Introduction

One starts with triangle ABC and some point Q not on the sides. Triangle ABC is then rotated about Q by 180^{o} to form a second triangle A*B*C*. So AQ = QA* etc. The point A* is now reflected in BC to produce a point X, B* is reflected in CA to produce Y and C* is reflected in AB to produce Z. Triangle XYZ turns out to be similar to triangle ABC, and circle XYZ we prove passes through H*, the orthocentre of triangle A*B*C* and has centre O, the circumcentre of ABC. This is the dual significance of circle XYZ. With respect to ABC its centre is the same as its circumcentre and with respect to A*B*C*, since it passes through H*, it is a Hagge circle. It is also the case that A, B, C, A*, B*, C* lie on a conic. It follows that A*, H*, X are collinear as are B*, H*, Y and C*, H*, Z. The following properties now follow from the fact that XYZH* is a Hagge circle. A point P_g exists such that $OO^*P_gH^*$ is a parallelogram. The isogonal conjugate of P_g, the point P is the point by which there is another construction of points X, Y, Z. To be precise if A*P, B*P, C*P meet the circumcircle of A*B*C* at D*, E*, F* respectively. Then if D*, E*, F* are reflected in B*C*, C*A*, A*B* respectively, the images U*, V*, W* lie on circle XYZ and are such that U*P, V*P, W*P meet the Hagge circle again at X, Y, Z respectively. Also shown in Fig. 1 is a line through P which is the axis of indirect similarity of triangles A*B*C* and XYZ, P being the centre of the similarity.

In the following sections we prove, using areal co-ordinates with ABC as triangle of reference, the features additional to what is already known about Hagge circles. See Bradley's 'Algebra of Geometry' p 419-420 for the co-ordinate geometry of Hagge circles.

2. Points A*, B*, C*, X and H*

Suppose Q has normalised co-ordinates Q(1 – u – v, u, v). Then, since Q is the midpoint of AA*, it follows that A* has co-ordinates A*(1 – 2u – 2v, 2u, 2v). Similarly B* has co-ordinates B*(2 – 2u – 2v, 2u – 1, 2v) and C* has co-ordinates C*(2 – 2u – 2v, 2u, 2v – 1).

The reflection of A* in BC is the point X with co-ordinates (x, y, z), where

$$x = 2u + 2v - 1, \tag{2.1}$$
$$y = 2u + (1 - 2u - 2v)(a^2 + b^2 - c^2)/a^2, \tag{2.2}$$
$$z = 2v + (1 - 2u - 2v)(c^2 + a^2 - b^2)/a^2. \tag{2.3}$$

The point H has normalised co-ordinates (x, y, z), where

$$x = (1/k)(a^2 + b^2 - c^2)(c^2 + a^2 - b^2), \tag{2.4}$$
$$y = (1/k)(b^2 + c^2 - a^2)(a^2 + b^2 - c^2), \tag{2.5}$$
$$z = (1/k)(c^2 + a^2 - b^2)(b^2 + c^2 - a^2), \tag{2.6}$$

where $k = 2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4$. The point H* lies on HQ and is such that HQ = QH*. It follows that H* has co-ordinates (x, y, z), where

$$x = 3 - 2u - 2v - (2/k)(a^2(b^2 + c^2) - (b^2 - c^2)^2), \tag{2.7}$$
$$y = 2u - 1 + (2/k)\, b^2(c^2 + a^2 - b^2), \tag{2.8}$$
$$z = 2v - 1 + (2/k)c^2(a^2 + b^2 - c^2). \tag{2.9}$$

It may now be proved by the determinantal method that A*, H*, X are collinear.

3. Points B*, C*, Y, Z and circle XYZ

B*, C* are the points on BQ, CQ respectively such that BQ = QB* and CQ = QC*. Their co-ordinates are therefore B*(2 – 2u – 2v, 2u – 1, 2v) and C*(2 – 2u – 2v, 2u, 2v – 1).

The point Y is the reflection of the point B* in CA and so has co-ordinates (x, y, z), where

$$x = (1/b^2)(a^2(2u - 1) + b^2(1 - 2v) + c^2(1 - 2u)), \quad (3.1)$$

$$y = 1 - 2u, \quad (3.2)$$

$$z = (1/b^2)(a^2(1 - 2u) + b^2(2u + 2v - 1) - c^2(1 - 2u)). \quad (3.3)$$

The point Z is the reflection of the point C* in AB and so has co-ordinates (x, y, z), where

$$x = (1/c^2)(a^2(2v - 1) + b^2(1 - 2v) + c^2(1 - 2u)) \quad (3.4)$$

$$y = (1/c^2)(a^2(1 - 2v) - b^2(1 - 2v) + c^2(2u + 2v - 1)), \quad (3.5)$$

$$z = 1 - 2v. \quad (3.6)$$

The equation of the circle XYZ is of the form

$$a^2yz + b^2zx + c^2xy + (lx + my + nz)(x + y + z) = 0, \quad (3.7)$$

where l, m, n depend upon the co-ordinates of X, Y, Z.

After inserting the co-ordinates of X, Y, Z in Equation (3.7) and solving the three equations obtained we find

$$l = m = n = a^2(1 - 2v)(2u - 1) + (2u + 2v - 1)(b^2(2v - 1) + c^2(2u - 1)). \quad (3.8)$$

It may now be checked that H* lies on circle XYZ. This means we have a Hagge circle and the simplicity of the equation representing it, in terms of the co-ordinates of Q, surely means that this way of determining Hagge circle is superior to any other.

The fact that A*, H*, X etc. are collinear shows that the points X, Y, Z correspond precisely to the points associated with Hagge's construction. We repeat from the Introduction how that construction is performed.

The following properties now follow from the fact that XYZH* is a Hagge circle. A point P_g exists such that OO*P_gH* is a parallelogram. The isogonal conjugate of P_g, the point P is the point by which there is another construction of points X, Y, Z. To be precise if A*P, B*P, C*P meet the circumcircle of A*B*C* at D*, E*, F* respectively. Then if D*, E*, F* are reflected in B*C*, C*A*, A*B* respectively, the images U*, V*, W* lie on circle XYZ and are such that U*P, V*P, W*P meet the Hagge circle again at X, Y, Z respectively. Also shown in Fig. 1 is a line through P which is the axis of indirect similarity of triangles A*B*C* and XYZ, P being the centre of the similarity

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