Article 13

Ex-points and eight intersecting circles

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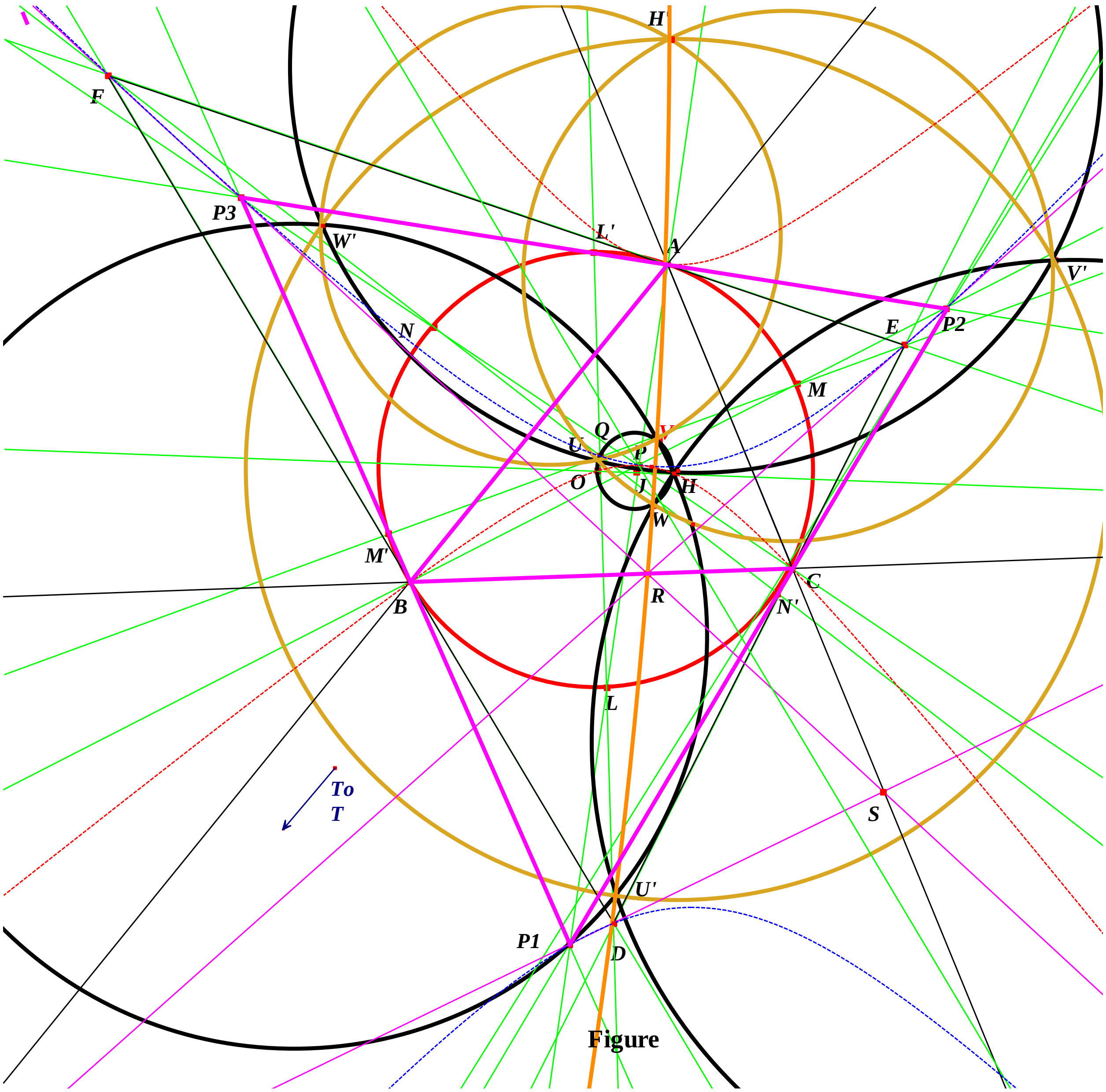

Figure

1. Introduction

In this article we use areal co-ordinates throughout. See Bradley [1, 2] for a review of how to use them. Incentres and ex-centres are well-known and if (a, b, c) are the co-ordinates of the incentre, then the three ex-centres I_1, I_2, I_3 have co-ordinates (– a, b, c), (a, – b, c), (a, b, – c) respectively. Less well-known, perhaps, are the ex-symmedian points D, E, F, which are the intersections of the tangents at the vertices of the circumcircle, D being the intersection of the tangents at B and C, with E, F similarly defined. These points can be seen in the Figure. Their co-ordinates are $(-a^2, b^2, c^2)$, $(a^2, -b^2, c^2)$, $(a^2, b^2 - c^2)$ respectively. It is also true that if P (l, m, n) is any point internal to a triangle ABC, then there are three associated ex-points $P_1(-l, m, n)$, $P_2(l, -m, n)$, $P_3(l, m, -n)$. We show how these points may be constructed using compass and straight edge only. It turns out that the construction is a development of the construction of the Hagge circle [3] of the point P, so we are able to draw four Hagge circles, one for each of P, P_1, P_2, P_3, all of which, of course, pass through the orthocentre H of triangle ABC. As a bonus there are in addition four other circles and the *CABRI II* geometry computer software package indicates that these other four circles also have a common point, labelled H' in the Figure. The additional four circles have equations that are considerably more complicated than those of the four Hagge circles, so we have been unable to find the co-ordinates of H'. However, it may be proved by inversion that these four circles do indeed have a common point. This proof has been supplied by David Monk [4] and I am extremely grateful to him for providing the proof.

We now describe the construction of the ex-points, starting from ABC and P. First draw the tangents at A, B, C to the circumcircle to provide D, E, F. Next draw the lines AP, BP, CP to meet the circumcircle at L, M, N respectively. Now draw DL, EM, FN to meet the circumcircle again at L', M', N' respectively. Note that these three lines are concurrent at a point Q, which we call the *generating point.* Now AL, BM', CN' are found to be concurrent at the point P_1, AL', BM, CN' are concurrent at P_2, and AL', BM', CN are concurrent at P_3. Section 2 provides the algebraic proof of these remarks. Q is called the generating point, because rather than start with P, one can start with Q, which must be internal to triangle DEF, and then reverse the construction, thus generating all 4 points P, P_1, P_2, P_3. We defer the description of the construction of the 8 circles until later in the text.

2. The ex-points P_1, P_2, P_3 and the generating point Q

For those unfamiliar with areal co-ordinates for points on the circumcircle Γ we give a brief account of what is required. The co-ordinates of a general point on Γ are $(-a^2t(1-t), b^2(1-t), c^2t)$, where t is a parameter that can take any value (including infinity). Thus the points A(1, 0, 0), B(0, 1, 0), C(0, 0,1) have parameters ∞, 0, 1 respectively and the equation of the circumcircle is

$$a^2yz + b^2zx + c^2xy = 0. \tag{2.1}$$

For checking purposes it is useful to have the equation of the chord of Γ joining points with parameters t and s. This is

$$\frac{x}{a^2} + \frac{sty}{b^2} + \frac{(1-s)(1-t)z}{c^2} = 0. \qquad (2.2)$$

Starting with P(l, m, n) we find the equation of AP to be ny = mz. This meets Γ, with Equation (2.1), at the point L with co-ordinates L(– a^2mn/(b^2n + c^2m), m, n). Similarly BP, CP meet Γ at M, N respectively with co-ordinates M(l, – b^2nl/(c^2l + a^2n), n) and N(l, m, – c^2lm/(a^2m + b^2l).

The tangents to Γ with Equation (2.1) at A, B, C are respectively $b^2z + c^2y = 0$, $c^2x + a^2y = 0$, $a^2y + b^2x = 0$.The point D, that is the intersection of the tangents at B and C, therefore has co-ordinates D(– a^2, b^2, c^2) and E, F respectively have co-ordinates E(a^2, – b^2, c^2), F(a^2, b^2, – c^2). It can now be seen that D, E, F are the ex-symmedians, as the symmedian point K has co-ordinates (a^2, b^2, c^2). The line DL has equation

$$(b^4n^2 - c^4m^2)x + a^2(b^2n^2y - c^2m^2z) = 0. \qquad (2.3)$$

This meets Γ again at the point L' with co-ordinates L'(a^2mn/(b^2n – c^2m), – m, n) Note that this differs from L simply by a change of sign of m (or n). We can now write down, by cyclic change, the co-ordinates of M' and N', which are M'(l, b^2nl/(c^2l – a^2n), – n) and N'(– l, m, c^2lm/(a^2m – b^2l)).

Now EM has equation that may be derived from that of DL by cyclic change and then the generating point Q is determined as DL^EM. The result is that Q has co-ordinates $(a^2(l^2(b^4n^2 + c^4m^2) - a^4m^2n^2), b^2\big(m^2(c^4l^2 + a^4n^2) - b^4n^2l^2\big), c^2(n^2(a^4m^2 + b^4l^2) - c^4l^2m^2)\big)$. It may now be checked that Q also lies on the line FN.

The equations of the lines AL', BM', CN' can now be determined and they are respectively

$$ny + mz = 0, lz + nx = 0, mx + ly = 0. \qquad (2.4)$$

We now have enough to determine the co-ordinates of the points P_1 = AL^ BM'^CN', P_2 = AL'^BM^CN' and P_3 = AL'^BM'^CN, and, as you will have guessed, these are the ex-points of P with co-ordinates P_1(– l, m, n), P_2(l, – m, n), P_3(l, m, – n). It may be checked that EP_2 and FP_3 intersect at a point R on BC, with S, T on CA, AB respectively defined similarly.

In the Figure we have chosen P to be the isogonal conjugate of the nine-point centre. This point generates the Hagge circle with OH as diameter.

3. A 9 point hyperbola

Using the standard 6 x 6 determinantal method to obtain the conic through P, its 3 ex-points and K the symmedian point we find the equation of a hyperbola

$$(b^4n^2 - c^4m^2)x^2 + (c^4l^2 - a^4n^2)y^2 + (a^4m^2 - b^4l^2)z^2 = 0. \qquad (3.1)$$

Its form immediately indicates that not only does this hyperbola pass through the points just mentioned, but it also passes through D, E, F the ex-symmedians. It may be checked that it also passes through the generating point Q. Thus, it passes through 9 key points of the figure. As P may be any point, we have the result, for example, that a 9 point hyperbola passes through the incentre, the ex-centres, the symmedian point, the ex-symmedians and the point Q, which in this case is the circumcentre O. It is a rectangular hyperbola, since the incentre and ex-centres form an orthocentroidal quartet. If P lies at O, then the hyperbola is also rectangular, since the orthocentre of triangle $O_1O_2O_3$ lies on the hyperbola.

4. The reflected points

The points in the Figure labelled U, V, W, U', V', W' are the reflections of the points L, M, N, L', M', N' in the sides BC, CA, AB, BC, CA, AB respectively. This is the same as in the construction of Hagge circles. The algebra for obtaining the co-ordinates of the reflected points is covered in detail in Bradley and Smith [3] and is not repeated here. We therefore state the co-ordinates of these points without proof and they are as follows:

$$U(a^2mn, -a^2mn + m(m+n)c^2, -a^2mn + n(m+n)b^2), \quad (4.1)$$

$$U'(-a^2mn, a^2mn + m(m-n)c^2, a^2mn + n(n-m)b^2), \quad (4.2)$$

$$V(-b^2nl + l(n+l)c^2,\ b^2nl,\ -b^2nl + l(n+l)c^2), \quad (4.3)$$

$$V'(b^2nl + l(l-n)c^2, -b^2nl, b^2nl + n(n-l)a^2), \quad (4.4)$$

$$W(-c^2lm + l(l+m)b^2, -c^2lm + m(l+m)a^2, c^2lm), \quad (4.5)$$

$$W'(c^2lm + l(l-m)b^2, c^2lm + m(m-l)a^2, -c^2lm). \quad (4.6)$$

5. The 8 circles

There is, of course a circle through any 3 of these points, but there are 8 special circles found by choosing one of U, U', one of V, V' and one of W, W'. First we consider the circle UVW. This is the Hagge circle generated by P. The method is to write down the general equation of a circle, which is

$$a^2yz + b^2zx + c^2xy + (x+y+z)(ux+vy+wz) = 0. \quad (5.1)$$

Then we insert the co-ordinates of U, V, W in turn, to provide three equations for the real numbers u, v, w. Their values are then substituted back in Equation (5.1). The result is

$$a^2(b^2+c^2-a^2)mnx^2 + b^2(c^2+a^2-b^2)nly^2 + c^2(a^2+b^2-c^2)lmz^2 - yz\left(a^4mn + l(b^2-c^2)(b^2n-c^2m)\right) - zx\left(b^4nl + m(c^2-a^2)(c^2l-a^2n)\right) - xy(c^4lm + n(a^2-b^2)(a^2m-b^2l)) = 0. \quad (5.2)$$

It may be checked that this circle passes through H, the orthocentre of ABC. The circle UV'W' has the equation derived from Equation (5.2) by changing the sign of l. It is therefore the Hagge circle generated by P_1. Similarly changes of sign of m, n in Equation (5.2) produce the Hagge circles of P_2, P_3 respectively.

The other four circles U'V'W', U'VW, UV'W, UVW' have equations that are extremely lengthy and complicated, and the algebra computer package *DERIVE* was unable to solve the simultaneous quadratics of any pair of them to show that they have the common point, labelled H' in the figure. However, David Monk [4] has kindly provided the following proof: Invert the figure in H so that circles UVW, UV'W', U'VW', U'V'W become the lines uvw, uv'w', u'vw', u'v'w. The resulting figure is a complete quadrilateral, and thus the four triangles u'v'w', u'vw, uv'w, uvw' have circumcircles that possess a common point h'. Inverting back gives the result required. This proof does not tell us anything about the dependence of H' on the point P. All that we can say from drawings is that the locus of H' as P moves on a line is a curve that is not a conic.

References

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2. C.J.Bradley, *The Algebra of Geometry*, Highperception, Bath (2007);
3. Christopher Bradley and Geoff C. Smith, *On a construction of Hagge*, Forum Geometricorum p231—247 (2006).
4. D. Monk (*private communication*).

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