Article: CJB/2011/125

# Properties of the Extangential Triangle

**Christopher Bradley**

Abstract: The external common tangents to the three ex-circles form the extangential triangle A'B'C'. Triangles ABC and A'B'C' are in perspective with perspector the orthocentre of the intouch triangle. The axis of perspective plays a prominent part in the configuration. Triangle A'B'C' and the orthic triangle are homothetic through the Clawson point.

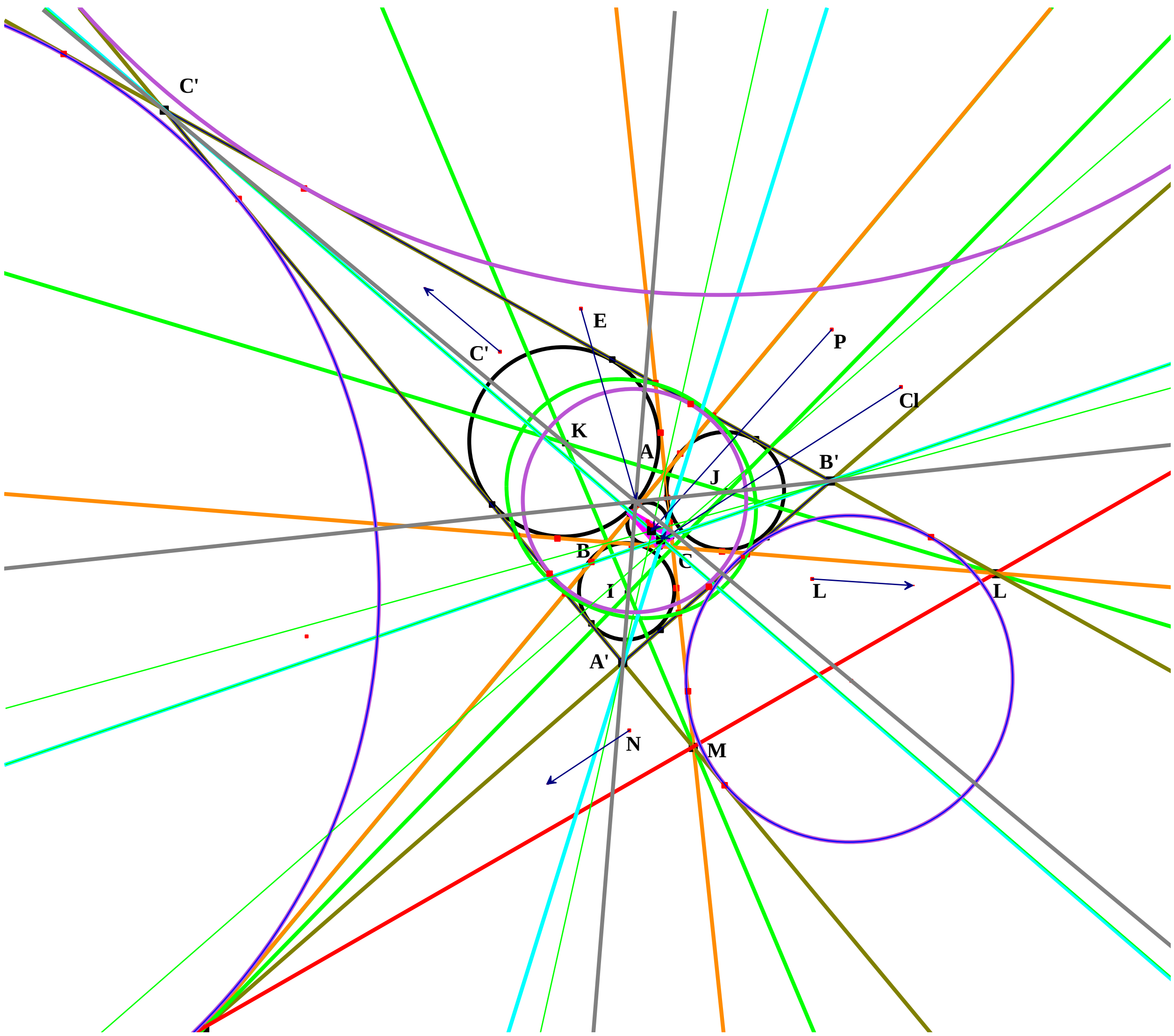


**Fig. 1**
**The extangential triangle**

## 1. Introduction

In the diagram we see a triangle ABC and its incircle and its three ex-circles. The external common tangents to the three pairs of excircles (other than the sides of ABC) form a triangle

A'B'C'. It is shown that ABC and A'B'C' are in perspective, with perspector the orthocentre of the intouch (or contact) triangle. This is the triangle consisting of the three points on the sides of ABC where the incircle touches them. The Desargues axis of perspective LMN is shown, and points L, M, N have the property that lines joining pairs of centres of the ex-circles pass through them. We also show that the triangles A'B'C' and the orthic triangle are homothetic with vertex of enlargement the Clawson point. Work is carried out using areal co-ordinates with ABC as triangle of reference.

## 2. The ex-circles and their points of contact with the sides of ABC

It is well known that the ex-circle opposite A has equation

$$(a + b + c)^2x^2 + (a + b - c)^2y^2 + (c + a - b)^2z^2 - 2(a + b - c)(c + a - b)yz + 2(a + b + c)(c + a - b)zx + 2(a + b + c)(a + b - c)xy = 0. \quad (2.1)$$

The ex-circles opposite B and C have similar equations that may be written down from Equation (2.1) by cyclic change of x, y, z and a, b, c.

The centres I, J, K of the ex-circles have co-ordinates I(– a, b, c), J(a, – b, c) and K(a, b, – c) so the line JK has equation

$$cy + bz = 0. \quad (2.2)$$

This meets BC at the point L with co-ordinates L(0, b, – c). Similarly M = CA^KI and N = AB^IJ have co-ordinates M(– a, 0, c) and N(a, – b, 0). L, M, N are collinear on the line with equation

$$x/a + y/b + z/c = 0, \quad (2.3)$$

which is the Desargues' axis of perspective of triangles ABC and IJK. We now tabulate, without details, the points of contact $L_1$, $L_2$, $L_3$ of the ex-circle opposite A with the sides BC, CA, AB respectively. These are $L_1$(0, c + a – b, a + b – c), $L_2$(b – c – a, 0, a + b + c) and $L_3$(c – a – b, a + b + c, 0). Similarly the points of contact $M_2$, $M_3$, $M_1$ of the ex-circle opposite B with the sides CA, AB, BC respectively are $M_2$(b + c – a, 0, a + b – c), $M_3$(a + b + c, c – a – b, 0), $M_1$(0,, a – b – c, a + b + c). And again the points of contact of the ex-circle opposite C with the sides AB, BC, CA respectively $N_3$(b + c – a, c + a – b, 0), $N_1$(0, a + b + c, a – b – c), $N_2$(a + b + c, 0, b – c – a).

## 3. The extangential triangle

With L as defined in section 1 we assert that the tangent from L to the ex-circle opposite B also touches the ex-circle opposite C. The proof is as follows:

Any line through L has an equation of the form x = k(cy + bz) for varying k. The value of k for this line to touch the excircle opposite B is k = – (b + c)/(bc) and the co-ordinates of the point of contact $J_1$ is (x, y, z), where

$$x = (a + b - c)(b + c)^2, \qquad (3.1)$$
$$y = - b^2(a + b + c), \qquad (3.2)$$
$$z = c^2(b + c - a). \qquad (3.3)$$

The line $LJ_1$ may now be seen to touch the ex-circle opposite C at the point $K_1$ with co-ordinates (x, y, z), where

$$x = (c + a - b)(b + c)^2, \qquad (3.4)$$
$$y = b^2(b + c - a), \qquad (3.5)$$
$$z = - c^2(a + b + c). \qquad (3.6)$$

The line $LJ_1K_1$ is now the side B'C' of the extangential triangle, with equation

$$bcx + (b + c)(cy + bz) = 0. \qquad (3.7)$$

Similarly the sides C'A' and A'B' have equations

$$cay + (c + a)(az + cz) = 0, \qquad (3.8)$$

and

$$abz + (a + b)(bx + ay) = 0. \qquad (3.9)$$

The lines with equations (3.8) and (3.9) meet at A' with co-ordinates (x, y, z), where

$$x = - a^2(a + b + c), \qquad (3.10)$$
$$y = b(c + a)(a + b - c), \qquad (3.11)$$
$$z = c(a + b)(c + a - b). \qquad (3.12)$$

The co-ordinates of B' and C' may be written down from equations (3.10) – (3.12) by cyclic change of x, y, z and a, b, c.

## 4. ABC and A'B'C' are in perspective

It may now be shown that AA', BB', CC' are concurrent at the point P with co-ordinates (x, y, z), where

$$x = a(b + c)/(b + c - a), \qquad (4.1)$$
$$y = b(c + a)/(c + a - b), \qquad (4.2)$$
$$z = c(a + b)/(a + b - c). \qquad (4.3)$$

P is the orthocentre of the intouch triangle, $X_{65}$ in Kimberling’s Encyclopaedia of Triangle Centres. It is interesting to note that line LMN is also the Desargue’s axis of perspective for these two triangles.

## 5. The orthic triangle of triangle ABC and triangle A'B'C' are homothetic

The third side of the orthic triangle has equation

$$(b^2 + c^2 – a^2)x = (c^2 + a^2 – b^2)y + (a^2 + b^2 – c^2)z. \quad (5.1)$$

The line B'C' with equation (3.7) meets this line at the point $(b^2 – c^2, – b^2, c^2)$, which is on the line at infinity. It follows that these two lines are parallel. This is also the case for the other two pairs of sides and hence the triangles are homothetic. The lines through corresponding pairs of vertices are concurrent at a point Cl with co-ordinates (x, y, z), where

$$x = a/(b^2 + c^2 – a^2),\ y = b/(c^2 + a^2 – b^2),\ z = c/(a^2 + b^2 – c^2). \quad (5.2)$$

The point Cl is called the Clawson point, $X_{19}$ in Kimberling's Encyclopaedia of Triangle Centres. Although John Clawson (1) studied this point in 1925, it was studied earlier by Lemoine.

*Reference*


1. Kimberling, C. "Encyclopedia of Triangle Centers: X(19) = Clawson Point." http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X19.



Flat 4,
Terrill Court,
12-14, Apsley Road,
BRISTOL BS8 2SP.