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Two Cevian Points Collinear with a Vertex and Thirteen Conics

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Abstract: Given triangle ABC, a point P on BC and lines LMP and UVP with L, U on AB and M, V on CA, then two Cevian points may be constructed collinear with vertex A. Two further Cevian points occur and the resulting configuration of points and lines results in the appearance of thirteen conics and two harmonic ranges.

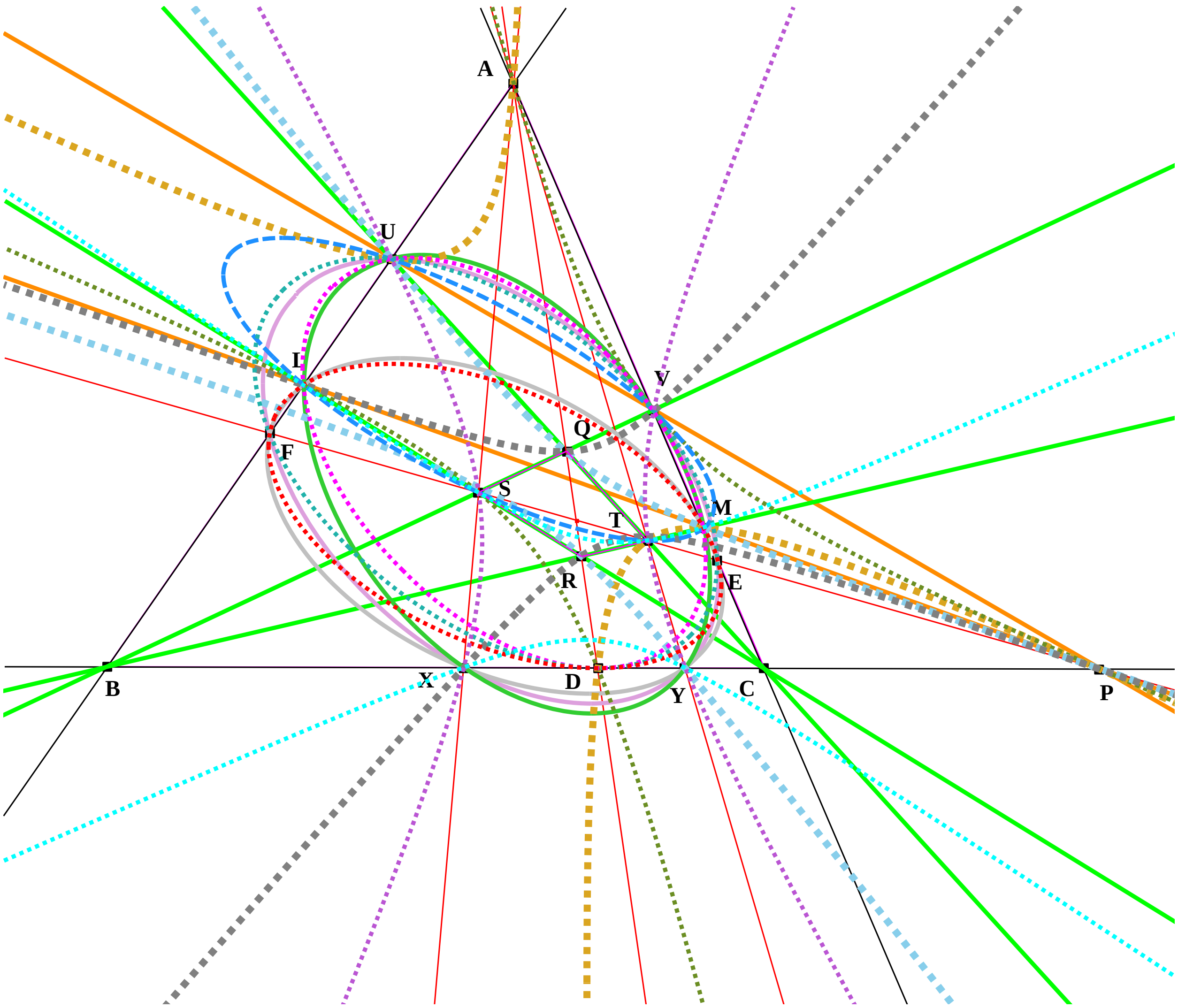

Fig. 1

Two Cevian points Q, R collinear with A and the thirteen resulting conics

1. Introduction

The construction of the above figure is as follows. Triangle ABC is given together with a point P on BC (not B or C). Two points U and L on AB are chosen and lines UVP, LMP are drawn with V, M on CA. The following lines are now drawn: BM, BV, CU, CL. These define four points, BV^CU = Q, BM^CL = R, BV^CL = S and BM^CU = T.

The line QR is shown to pass through A and the harmonic conjugate D of P relative to B and C. The line ST meets AB at F and CA at E and is shown to pass through P.
Lines AS, AT meet BC at X, Y respectively and X, Y are also harmonic conjugates of D, P.

The following thirteen conics are now shown to exist:
XYLMUV, XYLMEF, XYEFUV, UVLMST, PQRLVX, PQRMUY, DLSPVA, DMTPUA, LMUVD touching BC at D, UVEFD touching BC at D, LMEFD touching BC at D, STXYUV and STXYLM.

The four points Q, R, S, T are Cevian points and two of them Q, R are collinear with a vertex. This article highlights what to expect when a pair of Cevian points are collinear with a vertex, and at the same time, though we make no analysis of this context at present, when three related complete quadrilaterals exist with points B, C, P in common.

In what follows areal co-ordinates are used with ABC as triangle of reference. There appears to be no simplification to be had by using homogeneous projective co-ordinates.

2. The lines and points

We take P to have co-ordinates (0, p, 1 – p), M to have co-ordinates M(1 – m, 0, m) and V to have co-ordinates V(1 – v, 0, v). The equation of PLM is therefore

$$mpx + (m - 1)(p - 1)y + (m - 1)pz = 0, \qquad (2.1)$$

with L having co-ordinates L(– (p – 1)(m – 1), pm, 0). Similarly the equation of PUV is

$$vpx + (v - 1)(p - 1)y + (v - 1)pz = 0, \qquad (2.2)$$

with U having co-ordinates U(– (p – 1)(v – 1), pv, 0).

The equations of BV, BM, CU, CL are as follows:

BV: $vx + (v - 1)z = 0,$ (2.3)

BM: $mx + (m - 1)z = 0.$ (2.4)

CU: $(p - 1)(v - 1)y + pvx = 0,$ (2.5)

CL: $(p - 1)(m - 1)y + pmx = 0.$ (2.6)

The co-ordinates of the Cevian points Q, R, S, T may now be determined and are

Q = BV^CU:	((1 – v)(p – 1), vp, v(p – 1)),	(2.7)
R = BM^CL:	((1 – m)(p – 1), mp, m(p – 1)),	(2.8)
S = BV^CL:	((1 – v)(m – 1)(p – 1), pm(v – 1), v(m – 1)(p – 1)),	(2.9)
T= BM^CU:	((1 – v)(m – 1)(p – 1), pv(m – 1), m(v – 1)(p – 1)).	(2.10)

The equation of ST may now be obtained and is

$$p(2mv - m - v)x + (m - 1)(v - 1)(p - 1)y + (m - 1)(v - 1)pz = 0. \quad (2.11)$$

Result 1

(i) ST passes through P: when x = 0, y = p and z = (1 – p).

(ii) ST meets CA at E with co-ordinates E((1 – v)(m – 1), 0, 2mv – m – v).

(iii) ST meets AB at F with co-ordinates

$$F((1 - v)(m - 1)(p - 1), p(2mv - m - v), 0).$$

Result 2

(i) The equation of QR is

$$(p - 1)y = pz. \quad (2.12)$$

(ii) QR passes through A(1, 0, 0).

(iii) AQR meets BC at D(0, p, p – 1), the harmonic conjugate of P with respect to B and C.

(iv) AS meets BC at X with co-ordinates X(0, mp(v – 1), v(m – 1)(p – 1)). (2.13)

(v) AT meets BC at Y with co-ordinates Y(0, vp(m – 1), m(v – 1)(p – 1)). (2.14)

(vi) D is the harmonic conjugate of P with respect to X and Y. The proof is left to the reader.

3. The Thirteen Conics

The equation of any conic may be put in the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \quad (3.1)$$

Here a, b, c are not the side lengths, but constants to be determined. For each conic we give the values of a, b, c, f, g, h in terms of p, m, v.

Conic XYLMUV

$a = 2m^2p^2v^2(m + v - 2mv)$, $b = 2mv(1 - v)(m - 1)(p - 1)^2(2mv - m - v)$,
$c = 2mv(1 - v)(m - 1)p^2(2mv - m - v)$,
$f = p(p - 1)(2mv - m - v)(2m^2v^2 - 2mv(m + v) + m^2 + v^2)$, $g = - mvp^2(2mv - m - v)^2$,
$h = - p(p - 1)mv(2mv - m - v)^2$.

Conic XYLMEF

$a = 2m^2p^2v(2mv - m - v)$, $b = 2mv(m - 1)^2 p - 1)^2(v - 1)$, $c = 2mvp^2(m - 1)^2(v - 1)$,

$f = p(1 - p)(m - 1)(2m^2v^2 - 2mv(m + v) + m^2 + v^2)$, $g = mvp^2(m - 1)(3mv - 2m - v)$,
$h = mv(m - 1)p(p - 1)(3mv - 2m - v)$.

Conic XYUVEF
Exchange m and v in the expressions for conic XYLMEF.

Conic UVLMST
$a = 2mvp^2$, $b = 2(m - 1)(v - 1)(p - 1)^2$, $c = 2(m - 1)(v - 1)p^2$, $f = p(m - 1)(v - 1)(p - 1)$,
$g = p^2(2mv - m - v)$, $h = p(p - 1)(2mv - m - v)$.

Conic PQRVLX
$a = 4mvp^2(m - v)$, $b = 2v(v - 1)(m - 1)^2(p - 1)^2$, $c = 2m(1 - m)(v - 1)^2p^2$,
$f = p(p - 1)(m - v)(m - 1)(v - 1)$, $g = m(1 - v)p^2(mv - 2m + v)$,
$h = p(p - 1)v(m - 1)(mv + m - 2v)$.

Conic PQRUMY
Exchange m and v in the expressions for conic PQRVLX.

Conic DLSPVA
$a = 0$, $b = 2(m - 1)(v - 1)(p - 1)^2$, $c = 2p^2(1 - v)(m - 1)$, $f = 0$, $g = p^2v(1 - m)$,
$h = p(1 - p)m(1 - v)$.

Conic DMTPUA
Exchange m and v in the expressions for conic DLSPVA.

Conic LMUVD touching BC at D
$a = 2mvp^2$, $b = 2(m - 1)(v - 1)(p - 1)^2$, $c = 2(m - 1)(v - 1)p^2$, $f = 2p(1 - p)(v - 1)(m - 1)$,
$g = p^2(2mv - m - v)$, $h = p(p - 1)(2mv - m - v)$.

Conic UVEFD touching BC at D
$a = 2p^2v(2mv - m - v)$, $b = 2(m - 1)(v - 1)^2(p - 1)^2$, $c = 2p^2(m - 1)(v - 1)^2$,
$f = 2p(1 - p)(m - 1)(v - 1)^2$, $g = p^2(v - 1)(3mv - m - 2v)$, $h = p(p - 1)(v - 1)(3mv - m - 2v)$.

Conic LMEFD touching BC at D
Exchange m and v in the expressions for conic UVEFD.

Conic STXYUV
$a = 2p^2v^3(m - 1)(2mv - m - v)$, $b = 2mv(1 - v)^3(m - 1)(p - 1)^2$, $c = 2mvp^2(1 - v)^3(m - 1)$,
$f = p(p - 1)(v - 1)^2(2m^2v^2 - 2mv(m + v) + m^2 + v^2)$, $g = p^2v^3(m - 1)^2(v - 1)$,
$h = pv^3(m - 1)^2(p - 1)(v - 1)$.

Conic STXYLM
Exchange m and v in the expressions for conic STXYUV.

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