Article 12

Circles concentric with the Circumcircle

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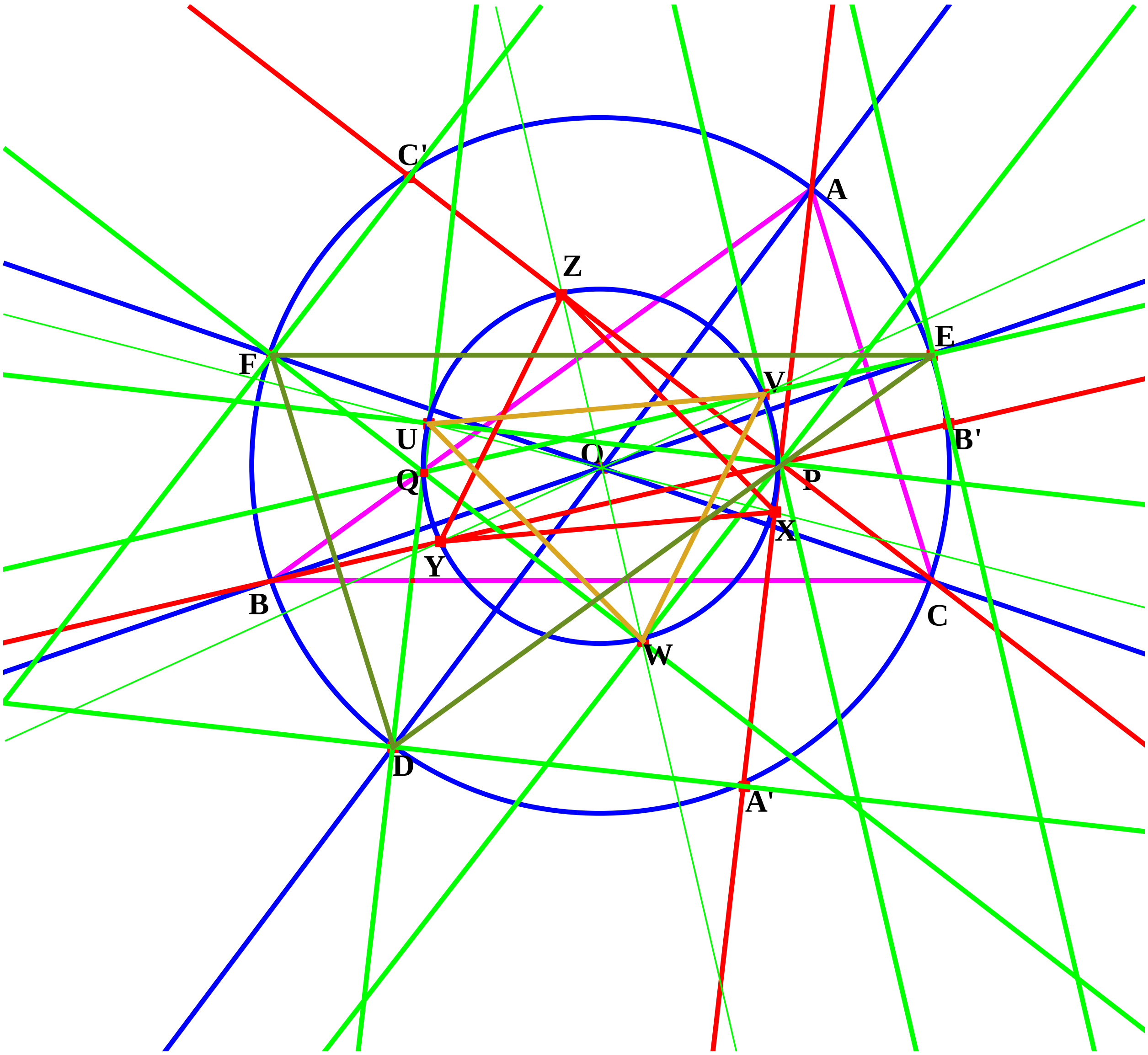

Figure

1. Introduction

Let ABC be a triangle with circumcircle S, centre O, and suppose P is a point not on the sides of ABC nor on S. In this short article we show that the circle, centre O, through P contains seven points U, V, W, X, Y, Z, Q with special properties. The figure above illustrates these properties which are a result of the following three theorems.

Theorem 1

Let AO, BO, CO meet S at D, E, F respectively. Let AP, BP, CP meet S at A', B', C' respectively. Draw lines l, m, n through P parallel to A'D, B'E, C'F respectively. Draw the line through D perpendicular to l to meet l at U. Define V, W similarly using E, m and F, n respectively. Then U, V, W lie on the circle Σ through P, centre O.

Theorem 2

Draw the diameters UOX, VOY, WOZ of Σ then X, Y, Z lie on AP, BP, CP respectively.

Theorem 3

AU, BV, CW are concurrent at a point Q lying on Σ.

In the proofs that follow we use Cartesian co-ordinates.

2. Proof of Theorem 1

Take O to be the origin, S to be the unit circle, with equation $x^2 + y^2 = 1$, and A to have co-ordinates $((1 - a^2)/(1 + a^2), 2a/(1 + a^2))$, with B and C having parameters b, c respectively. Let P have co-ordinates (c, 0), $c \neq 0, 1$.

The equation of AP is therefore

$$2ax + y(a^2(c + 1) + c - 1) = 2ac. \qquad (2.1)$$

This line meets S again at the point A' with co-ordinates (x, y), where

$$x = \{a^2(c + 1)^2 - (c - 1)^2\}/\{a^2(c + 1)^2 + (c - 1)^2\}, \qquad (2.2)$$

$$y = \{2a(c^2 - 1)\}/ \{a^2(c + 1)^2 + (c - 1)^2\}. \qquad (2.3)$$

Since AOD is a diameter, the co-ordinates of D are the negatives of those of A and thus the equation of A'D is

$$x(a^2(c + 1) + (c - 1)) - 2ay = a^2(c + 1) - (c - 1). \qquad (2.4)$$

The line l is parallel to this through P and therefore has equation

$$x(a^2(c + 1) + (c - 1)) - 2ay = c(a^2(c + 1) + (c - 1)). \qquad (2.5)$$

The equation of the line through D perpendicular to A'D is

$$2ax + y(a^2(c + 1) + (c - 1)) = - 2ac. \qquad (2.6)$$

The lines with equations (2.5) and (2.6) meet at the point U with co-ordinates (x, y), where

$$x = \{c(a^4(c + 1)^2 + 2a^2(c^2 – 3) + (c – 1)^2)\}/\{a^4(c + 1)^2 + 2a^2(c^2 + 1) + (c – 1)^2\}, \qquad (2.7)$$

$$y = – \{4ac(a^2(c + 1) + (c – 1)\}/ \{a^4(c + 1)^2 + 2a^2(c^2 + 1) + (c – 1)^2\}. \qquad (2.8)$$

The sum of the squares of these co-ordinates is c^2, so U lies on the circle Σ, centre O, radius c, which passes through P. Similarly V, W lie on Σ.

3. Proof of Theorems 2 and 3

Since UOX is a diameter of Σ the co-ordiantes of X are the negatives of those of U and it is soon checked that X lies on AP with Equation (2.1).

The line DU with equation (2.6) meets Σ again at the point Q with co-ordiantes (– c, 0) and since its co-ordinates are independent of Q also lies on EV and FW. It also lies on the diameter PO.

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