Article: CJB/2010/119

The Nine-point Conic and a Pair of Parallel Lines

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Abstract: An affine transformation of the nine-point circle leads to the nine-point conic. The orthocentre transforms into a general point P which is collinear with the centre of the nine-point conic and the centroid. We prove that the polar of P with respect to the nine-point conic is parallel to the Desargues' axis of perspective of the triangle and the image of the orthic triangle.

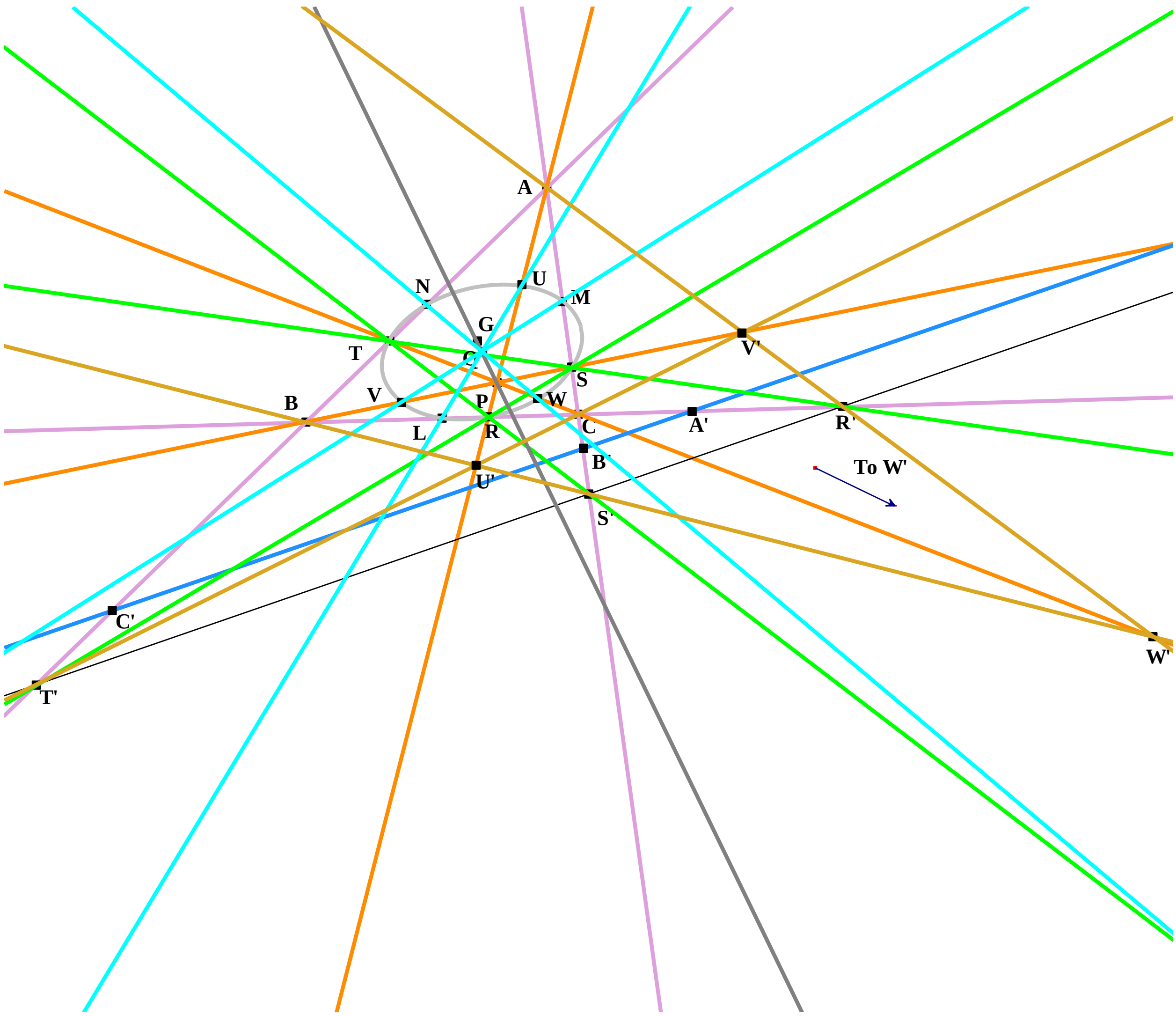

Fig.1
The nine-point conic

1. Introduction

A triangle ABC is given and also a general point P not on the sides of ABC. The Cevians AP, BP, CP meet the sides of ABC at points R, S, T. The midpoints of the sides of ABC are denoted by L, M, N. The conic RSTLMN is the nine-point conic Σ and it passes through the midpoints U, V, W of AP, BP, CP respectively. The points R', S', T' on BC, CA, AB are the harmonic conjugates of R, S, T respectively. It is verified that this is the Desargues' axis of perspective of triangles ABC and PQR (as is always the case). The polar of P with respect to the conic Σ meets the sides BC, CA, AB respectively at A', B', C'. It is proved that A'B'C' is parallel to R'S'T'. The point U' = AR^BS' is defined, with V' and W' similarly defined. The Desargues' axis of perspective of triangle ABC and U'V'W' is also the line R'S'T. We prove results, as necessary, using areal co-ordinates with ABC as triangle of reference.

2. The conic LMNRST

The points L, M, N have co-ordinates L(0, 1, 1), M(1, 0, 1), N(1, 1, 0). Suppose P has co-ordinates (l, m, n). Then R, S, T have co-ordinates R(0, m, n), S(l, 0, n), T(l, m, 0). It may now be shown that the conic Σ passing through these six points has equation

$$mnx^2 + nly^2 + lmz^2 - l(m + n)yz - m(n + l)zx + n(l + m)xy = 0. \qquad (2.1)$$

Σ meets AP again at the point U with co-ordinates U(2l + m + n, m, n). Similarly it passes through V(l, l + 2m +n, n) and W(l, m, l + m + 2n). It is easy to see that U, V, W are the midpoints of AP, BP, CP respectively. This does, of course follow from the fact that the nine-point conic arises from the nine-point circle by means of an affine transformation, which preserves ratios of line segments.

3. The centre Q of Σ and the collinearity of P, Q, G where G is the centroid of ABC

The centre Q of the conic Σ has co-ordinates Q(2l + m + n, l + 2m + n, l + m + 2n). It may be checked that G, Q, P are collinear and that GQ = (1/3)QP, which reflects the fact that on the Euler line GN = (1/3)NH, where N is the centre of the nine-point circle and H is the orthocentre.

4. The polar of P with respect to Σ

The equation of the polar of P with respect to Σ is

$$mn(m + n)x + nl(n + l)y + lm(l + m)z = 0. \qquad (4.1)$$

This meets BC at A' with co-ordinates A'(0, – m(l + m), n(n + l)). Points B', C' on CA, AB respectively have co-ordinates B'(l(l + m), 0, – n(m + n)), C'(– l(n + l), m(m + n), 0).

5. The points R', S', T' and the result that A'B'C' is parallel to R'S'T'

The equation of the line ST is

$$nly + lmz = mnx. \qquad (5.1)$$

This meets the side BC at the point R' with co-ordinates R'(0, – m, n). Points S', T' on CA, AB respectively have co-ordinates S'(l, 0, – n), T'(– l, m, 0). The points R', S', T' are harmonic conjugate of R, S, T respectively with respect to B, C and C, A and A, B. R'S'T' is, of course a straight line, the transversal associated with the Cevian point P, and again is well-known to be the Desargues' axis of perspective of triangles ABC and RST.

The equation of R'S'T' is found to be

$$mnx + nly + lmz = 0. \qquad (5.2)$$

The point of intersection of A'B'C' and R'S'T' has co-ordinates (l(m – n), m(n – l), n(l – m)) which lies on the line at infinity x + y + z = 0 and hence A'B'C' and R'S'T' are parallel.

6. The points U', V', W'

The lines AR and BS' meet at the point U' with co-ordinates U'(– l, m, n). Similarly BS and CT' meet at V'(l, – m, n) and CS and AR' meet at W'(l, m, – n). The equation of the line V'W' is therefore ny + mz = 0, which meets BC at R'. It follows that R'S'T' is also the Desargues' axis of perspective of triangles ABC and U'V'W'.

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