Article: CJB/2010/118

# Two Triangles in Perspective and inscribed in a Conic

**Christopher Bradley**

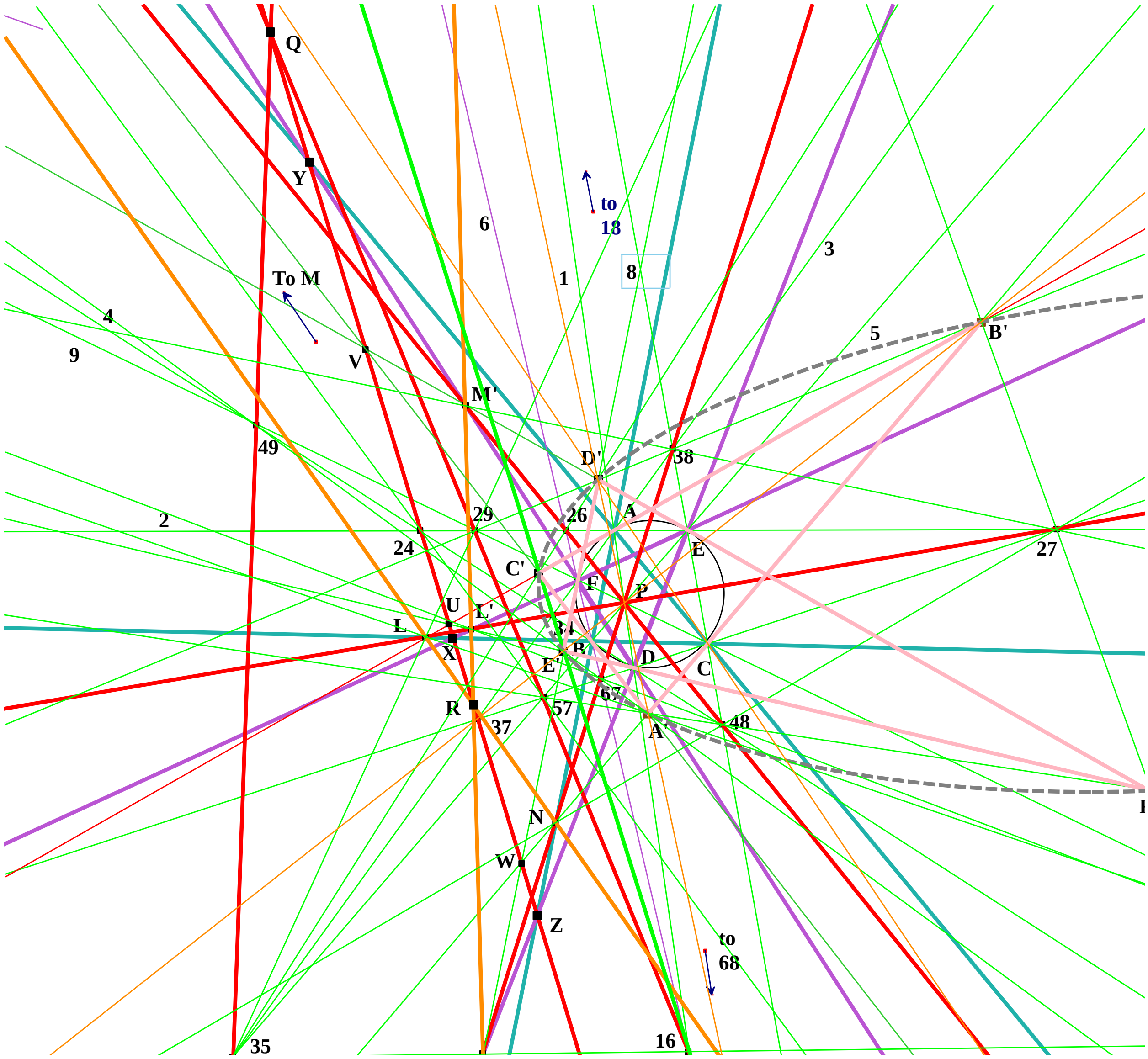


**Fig. 1**

**6 Pascal lines, 2 Steiner points, a Conic and many Points and Lines**

Abstract: Given two triangles ABC, DEF in perspective at P and inscribed in a conic Σ, a configuration is produced consisting of the 6 Pascal lines and 2 Steiner points arising. One of the Pascal lines has further properties, as it is not only a Pascal line, but also a Desargues' axis of perspective and also the polar of P with respect to Σ. Further properties involving the nine lines AD, AE, AF, BD, BE, BF, CD, CE, CF and the tangents at the six vertices are deduced leading

to a conic, further properties of the key Pascal line and numerous sets of collinear points and concurrent lines.

## 1. Introduction

Given a triangle ABC and a point P not on its sides lines AP, BP, CP are drawn to meet the circumconic Σ of ABC at points D, E, F respectively. The nine lines AD, AE, AF, BD, BE, BF, CD, CE, CF are drawn and in Fig. 1 these lines are marked 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively. The six Pascal lines resulting from these nine lines are now drawn. These are Pascal line A: AE^BD, AF^CD, BF^CE, which is also the Desargues' axis of perspective of triangles ABC and DEF; Pascal line B: AE^CD, AF^BD, BE^CF; Pascal line C: AD^BE, AF^CE, BF^CD; Pascal line D: AF^BE, AD^CE, BD^CF; Pascal line E: AE^BF, AD^CF, BD^CE; Pascal line F: AD^BF, AE^CF, BE^CD. We show that three of these pass through P and three through a point Q on Pascal line A. P and Q are therefore the 2 Steiner points of the configuration, which are common points of intersection of three Pascal lines. (In a complete configuration there are 60 Pascal lines and 20 Steiner points.)

The point of this article is to study the interrelationship of the Pascal lines with the tangents to Σ at the vertices of the triangles ABC and DEF or if you like to study the interrelationship of the two properties: (i) ABC and DEF are inscribed in Σ and (ii) ABC and DEF are in perspective. First however we introduce our labelling of points in the configuration, which at first sight may appear confusing, but actually is critical for describing the configuration. We label AE^BD as 24 as AE is line 2 and BD is line 4 (see above); similarly AF^CD is point 37 and so on. Points in Fig.1 are labelled by this method.

Now we describe some more key points, where $t_A$ for example means the tangent at A: (i) L is BC^$t_A$ and (as we shall prove) also lies on 35 29 and 34 P 27, (ii) M is CA^$t_B$ and also lies on 24 57 and 26 P 48, (iii) N is AB^$t_C$ and also lies on 48 27 and 67 P 38. We know from elementary theory that L, M, N are collinear. Now (iv) L' is EF^$t_D$ and also lies on 67 48 and 34 P 27; (v) M' is FD^$t_E$ and also lies on 38 27 and 26 P 48; (vi) N' is DE^$t_F$ and also lies on 35 16 and 67 P 38. Again L', M', N' are collinear. We prove that LMN and L'M'N' intersect at a point R on the key Pascal line A.

Now $t_B$ and $t_C$ meet at A' which also lies on 48 57, $t_C$ and $t_A$ meet at B' which also lies on 38 29, $t_A$ and $t_B$ meet at C' which also lies on 35 26, $t_E$ and $t_F$ meet at D' which also lies on 38 29 B', $t_F$ and $t_D$ meet at E' which also lies on 34 16 C' and $t_D$ and $t_E$ meet at F' which also lies on 48 57 A'.

The following collinearities also occur A'E' passes through 67 and 49, B'F' passes through 27 and 18 and C'D' passes through 26 and 35.

In the Sections that follow we give a co-ordinate treatment of the configuration using homogeneous projective co-ordinates with ABC as triangle of reference and P as unit point.

## 2. Points D, E, F and the nine lines AD, AE, AF, BD, BE, BF, CD, CE, CF

We take P to be the unit point (1, 1, 1) and the circumconic Σ to have equation

$$pyz + qzx + rxy = 0. \qquad (2.1)$$

D is the point of intersection of the line y = z with the conic Σ and therefore has co-ordinates (– p, q + r, q + r). Similarly E and F have co-ordinates E(r + p, – q, r + p) and F(p + q, p + q, – r).

Simple computations now provide the equations of the nine lines, which are:

| | | |
|---|---|---|
| AD: | $y = z$, | (2.2) |
| AE: | $(r + p)y + qz = 0$, | (2.3) |
| AF: | $ry + (p + q)z = 0$, | (2.4) |
| BD: | $(q + r)x + pz = 0$, | (2.5) |
| BE: | $z = x$, | (2.6) |
| BF: | $rx + (p + q)z = 0$, | (2.7) |
| CD: | $py + (q + r)x = 0$, | (2.8) |
| CE: | $(r + p)y + qx = 0$, | (2.9) |
| CF: | $x = y$. | (2.10) |

## 3. The 18 points and the 6 Pascal lines

We give the co-ordinates of the three points on a Pascal line and follow it up with the equation of the Pascal line itself. This is done six times, once for each Pascal line.

| | | |
|---|---|---|
| 24: | $(p(r + p), q(q + r), -(r + p)(q + r))$, | |
| 37: | $(p(p + q), -(p + q)(q + r), r(q + r))$, | |
| 68: | $(-(p + q)(r + p), q(p + q), r(r + p))$. | |
| Pascal line A: | $(q + r)x + (r + p)y + (p + q)z = 0$. | (3.1) |

| | | |
|---|---|---|
| 27: | $(pq, -q(q + r), (r + p)(q + r))$, | |
| 34: | $(rp, (p + q)(q + r), -r(q + r))$, | |
| 59: | (1, 1, 1). | |
| Pascal line B: | $(q + r)x = ry + qz$. | (3.2) |

| | |
|---|---|
| 15: | (1, 1, 1), |
| 38: | $((p + q)(r + p), -q(p + q), qr)$, |
| 67: | $(-p(p + q), (p + q)(q + r), rp)$. |

Pascal line C: $(p + q)z = qx + py$. (3.3)

35: $(r, -p-q, r)$,
18: $(-r-p, q, q)$,
49: $(p, p, -q-r)$.
Pascal line D: $qx + ry + pz = 0$. (3.4)

26: $((p + q)(r + p), qr, -r(r + p)$,
19: $(1, 1, 1)$,
48: $(-p(r + p), pq, (r + p)(q + r))$.
Pascal line E: $(r + p)y = rx + pz$. (3.5)

16: $(-p-q, r, r)$,
29: $(q, q, -r-p)$,
57: $(p, p, -q-r)$.
Pascal line F: $rx + py + qz = 0$ (3.6)

The three Pascal lines B, C and E all pass through P(1, 1, 1). The three Pascal lines A, D, F all pass through a point Q with co-ordinates $(p^2 - qr, q^2 - rp, r^2 - pq)$. P and Q are the two Steiner points. It may now be verified that Pascal line A is the polar of P with respect to the conic Σ.

## 4. The lines EF, FD, DE and the points X, Y, Z

From the co-ordinates of D, E, F in Section 2 we may obtain the equations of the lines EF, FD, DE. That of EF is

$$-px + (r + p)y + (p + q)z = 0. \quad (4.1)$$

Those of FD and DE may be obtained from equation (4.1) by cyclic permutation of x, y, z and p, q, r.

The point X = BC^EF then has co-ordinates $X(0, -p-q, r+p)$. Similarly Y = CA^FD has co-ordinates $Y(p + q, 0, -q-r)$ and Z = AB^ED has co-ordinates $Z(-r-p, q + r, 0)$. It may now be checked that X, Y, Z all lie on Pascal line A, verifying that the Desargues' axis of perspective coincides with Pascal line A.

## 5. The tangents at A, B, C, D, E, F and the points U, V, W

The tangents at A, B, C have equations:

$t_A$: $qz + ry = 0$, (5.1)
$t_B$: $rx + pz = 0$, (5.2)

$t_C$: $py + qx = 0.$ (5.3)

The tangents at D, E, F have equations:

$t_D$: $(q + r)^2x + pqy + rpz = 0,$ (5.4)

$t_E$: $pqx + (r + p)^2y + qrz = 0,$ (5.5)

$t_F$: $rpx + qry + (p + q)^2z = 0.$ (5.6)

The tangent at A meets the tangent at D at the point U with co-ordinates U(p(q – r), – q(q + r), r(q + r)). Similarly points V and W have co-ordinates V(p(r + p), q(r – p), – r(r + p)) and W(– p(p + q), q(p + q), r(p – q)). It may now be checked that U, V, W also lie on Pascal line A.

## 6. The lines LMN and L'M'N' and the point R

The line BC and the tangent at A meet at the point L with co-ordinates (0, q, – r). It may now be checked that L lies on the lines 35 29 and 34 P 27 with equations

$$px + ry + qz = 0, \quad (6.1)$$

and

$$ry + qz = (q + r)x. \quad (6.2)$$

Similarly M and N have co-ordinates M(– p, 0, r) and N(p, – q, 0). The line LMN clearly has the equation

$$x/p + y/q + z/r = 0. \quad (6.3)$$

The line EF and the tangent at D have equations

$$(r + p)y + (p + q)z = px, \quad (6.4)$$

and

$$(q + r)^2x + pqy + rpz = 0. \quad (6.5)$$

These lines meet at the point L' with co-ordinates L'(p(q – r), – rp – q(q + r), pq + r(q + r)). It may now be checked that L' lies on the lines D 67 48 and 34 P 27 which have equations

$$rp(p + q)z + pq(r + p)y + (pq^2 + pr^2 + rq^2 + qr^2 + pqr)x = 0, \quad (6.6)$$

and

$$ry + qz – (q + r)x = 0. \quad (6.7)$$

The co-ordinates of M' and N' may be written down from the co-ordinates of L' by cyclic change of x, y, z and p, q, r. The equation of the line L'M'N' may now be deduced and is

$$p(q^2 + qr + r^2)x + q(r^2 + rp + p^2)y + r(p^2 + pq + q^2)z = 0. \quad (6.8)$$

The lines LMN and L'M'N' meet at the point R with co-ordinates $R(p^2(q – r), q^2(r – p), r^2(p – q))$. It may now be checked that R also lies on Pascal line A.

## 7. The points A', B', C', D', E', F ' and the conic through these six points

The tangents at B and C with equations given in Section 5 and the line 57 48 with equation

$$(pq + r(q + r))x + p(p + r)y + prz = 0 \tag{7.1}$$

meet at the point A' with co-ordinates (– p, q, r). Similarly the tangents at A and C and the line 29 38 meet at the point B' with co-ordinates (p, – q, r). And similarly the tangents at A and B and the line 35 26 meet at the point C' with co-ordinates (p, q, – r).

The tangents at E and F given in Section 5 and the line 38 29 with equation

$$pqx + (p^2 + pr + qr)y + q(p + q)z = 0 \tag{7.2}$$

meet at the point D' with co-ordinates (x, y, z), where

$$x = -(p^2 + p(q + r) + 2qr), \tag{7.3}$$

$$y = q(p + q - r), \tag{7.4}$$

$$z = r(p - q + r). \tag{7.5}$$

The co-ordinates of E' and F' may be written down from equations (7.3) – (7.5) by cyclic change of x, y, z and p, q, r. The point E' is the intersections of the tangents at F and D and the line 34 16. The point F' is the intersection of the tangents at D and E and the lie 48 57.

It can now be shown that the points A', B', C', D', E,' F' lie on a conic with equation of the form

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \tag{7.6}$$

with

$$u = 2pqr(q + r), \tag{7.7}$$

$$v = 2pqr(r + p), \tag{7.8}$$

$$w = 2pqr(p + q), \tag{7.9}$$

$$f = p(p^2q + p^2r + q^2r + q^2p + r^2p + r^2q), \tag{7.10}$$

$$g = q(p^2q + p^2r + q^2r + q^2p + r^2p + r^2q), \tag{7.11}$$

$$h = r(p^2q + p^2r + q^2r + q^2p + r^2p + r^2q). \tag{7.12}$$

## 8. Three unexplained collinearities

The equation of C'D' is

$$rpx + r(r + p)y + (p^2 + pq + pr)z = 0. \tag{8.1}$$

It may now be shown that points 26 and 35 lie on this line. Similarly B'F' passes through points 18 and 27, and A'E' passes through the points 49 and 67.

Flat 4
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP.