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The Geometry of the Brocard Axis and Associated Conics

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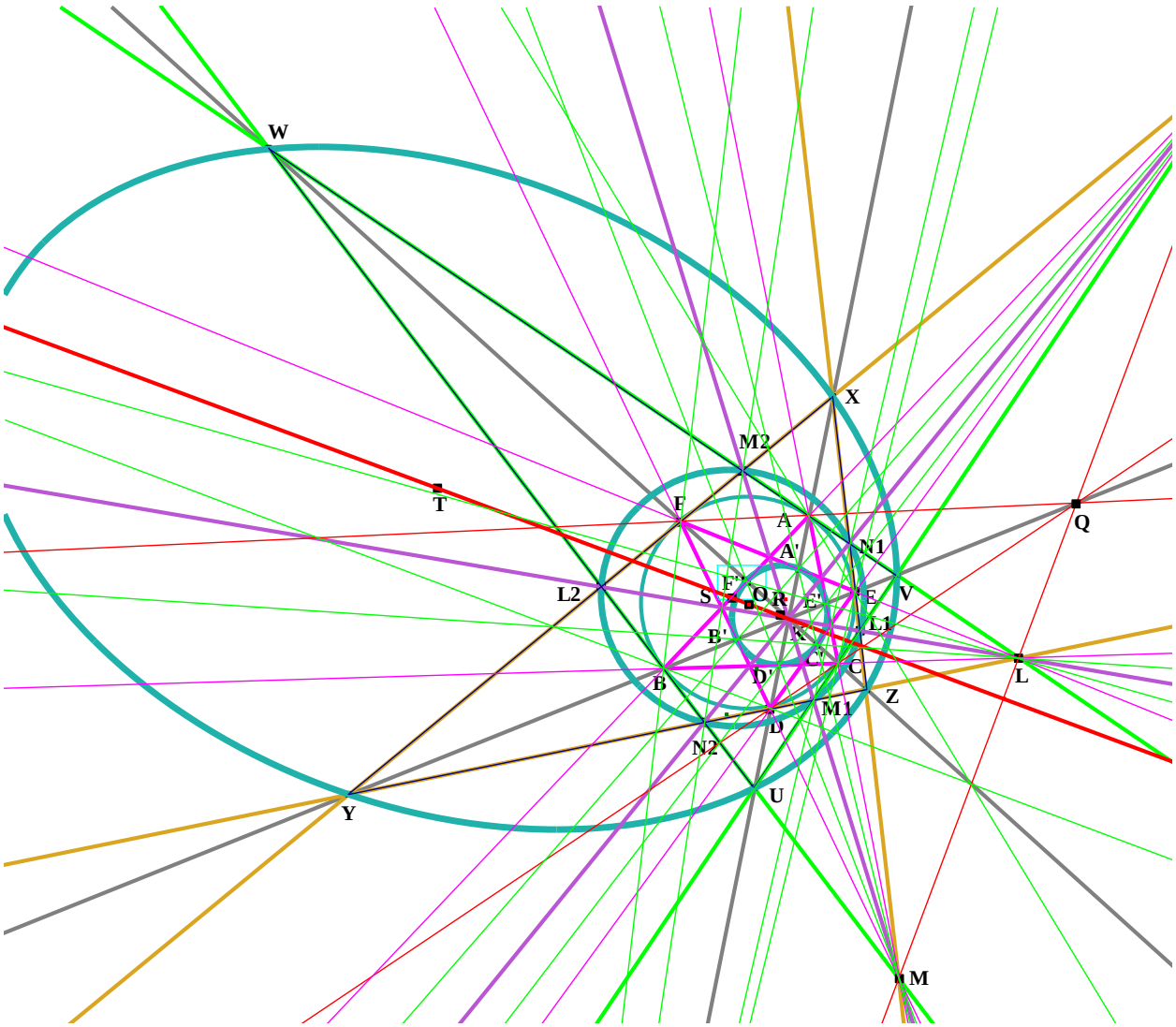

Fig.1
The Brocard axis and its conics

Abstract: From a triangle and its symmedian point K and its circumcentre O, various triangles, the Brocard inellipse and two further conics are constructed having their centres on the Brocard axis KO. Two porisms are defined and a special line is identified. Most, if not all, of the content of this article is known and indeed should be known by all geometers.

1. Introduction

Triangle ABC and its symmedian point K are given. Points D, E, F are the intersections of the lines AK, BK, CK with the circumcircle Σ. The tangents are drawn at A, B, C creating a triangle UVW, where U, V, W lie on AK, BK, CK respectively. Tangents are also drawn at D, E, F creating a second triangle XYZ and X, Y, Z also lie on AK, BK, CK respectively. It follows that triangles ABC, DEF are in perspective, vertex K and triangles UVW and XYZ are in perspective, vertex also K. We prove the result that X, W, Y, U, Z, V lie on a conic, centre T that lies on the Brocard axis KO.

The nine intersections of triangles UVW and XYZ lie six on a second conic, whose centre S lies on the Brocard axis KO, and the remaining three on a line LMN. The points on the conic are L_1, L_2 on a line through L, M_1, M_2 on a line through M and N_1, N_2 on a line through N. The line LMN is the polar of K with respect to Σ and the points on it have many properties. L, for example, lies on the lines BC, VW, YZ, EF, L_1L_2, B'C', E'F' and the point labelled Q lies on AF, BE, CD showing that it is the vertex of perspective of triangles ABC and FED. Such a perspective is called a reverse perspective and the triangles ABC and DEF are, in fact, in triple reverse perspective, ABC being in perspective with DFE, EDF and FED. The vertices of these perspectives all lie on line LMN. Further lines passing through Q are L_2N_1, N_2L_1, XW, YV, ZU. This shows that triangles XYZ and UVW besides being in direct perspective with vertex K are also in reverse perspective with vertex Q. In fact, they too are in triple reverse perspective, with vertices the same as the reverse perspectives of ABC and DEF.

The intersections of triangles ABC and DEF do not appear in Fig. 1, but six of them also lie on a conic and three of them, such as AB and DE meet at the point N on the polar line, AC and DF meet at M and BC and EF at L. Thus LMN is the Desargues' line of perspective of the triangles ABC and DEF.

Also if the lines AD, BE, CF meet BC, CA, AB at D', E', F' respectively and AD, BE, CF meet EF, FD, DE at A', B', C' respectively, then A', B', C', D', E', F' lie on a conic touching the sides of ABC and DEF. This is sometimes known as the Brocard inellipse. Its centre R lies on the Brocard axis KO. Thus one has two triangles ABC and DEF with vertices on Σ and sides touching the Brocard inellipse, and hence a porism is created, usually known as the Brocard porism. It is also the case that triangles XYZ and DEF have their vertices on the conic centre T, which by the initial construction have their sides touching Σ and hence we have a second porism, which I call the *outer Brocard porism*.
In what follows we use areal co-ordinates with ABC as triangle of reference to establish many of the above results, some, however, being left for the reader.

2. The points D, E, F and U, V, W

The point D is the intersection of the line AK with the circumcircle Σ. Since K has co-ordinates (a^2, b^2, c^2), the line AK has equation $c^2y = b^2z$. The circumcircle Σ has equation

$$a^2yz + b^2zx + c^2xy = 0. \tag{2.1}$$

It follows that D has co-ordinates $D(-a^2, 2b^2, 2c^2)$. Similarly E has co-ordinates $E(2a^2, -b^2, 2c^2)$ and F has co-ordinates $F(2a^2, 2b^2, -c^2)$.

The tangents at A, B and C have equations $b^2z + c^2y = 0$, $c^2x + a^2z = 0$ and $a^2y + b^2x = 0$ respectively. Those at B and C meet at the point U, which therefore has co-ordinates $U(-a^2, b^2, c^2)$. Similarly V and W have co-ordinates $V(a^2, -b^2, c^2)$ and $W(a^2, b^2, -c^2)$.

3. The points X, Y, Z and the conic UVWXYZ

The tangent to Σ at D has equation

$$4x/a^2 + y/b^2 + z/c^2 = 0. \tag{3.1}$$

And the tangents to Σ at E and F have equations

$$x/a^2 + 4y/b^2 + z/c^2 = 0, \tag{3.2}$$
$$x/a^2 + y/b^2 + 4z/c^2 = 0. \tag{3.3}$$

The tangents at E and F meet at X, which therefore has co-ordinates $X(5a^2, -b^2, -c^2)$. Similarly Y and Z have co-ordinates $Y(-a^2, 5b^2, -c^2)$ and $Z(-a^2, -b^2, 5c^2)$.

We can now determine the equation of the conic that passes through the six points X, Y, Z, U, V, W. This has the form

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \tag{3.4}$$

where $u = 2b^4c^4$, $v = 2c^4a^4$, $w = 2a^4b^4$, $f = 3a^4b^2c^2$, $g = 3a^2b^4c^2$, $h = 3a^2b^2c^4$.

In terms of u, v, w, f, g, h the centre of the conic with equation (3.4) has co-ordinates (x, y, z), where

$$\begin{aligned} x &= vw - gv - hw - f^2 + fg + hf, \\ y &= wu - hw - fu - g^2 + gh + fg, \\ z &= uv - fu - gv - h^2 + hf + gh. \end{aligned} \tag{3.5}$$

The centre T of conic XYZUVW therefore has co-ordinates (x, y, z), where

$$x = a^2(3b^2 + 3c^2 - 5a^2),$$

$$y = b^2(3c^2 + 3a^2 - 5b^2), \qquad (3.6)$$
$$z = c^2(3a^2 + 3b^2 - 5c^2).$$

4. The Brocard axis

The Brocard axis is the line KO, where K is the symmedian point (a^2, b^2, c^2) and O is the circumcentre of triangle ABC $(a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2))$, and it therefore has equation

$$(b^2 - c^2)x/a^2 + (c^2 - a^2)y/b^2 + (a^2 - b^2z/c^2 = 0. \qquad (4.1)$$

It may now be checked that T lies on the Brocard axis.

5. The sides of the triangles UVW and XYZ and the line LMN

VW has equation

$$c^2y + b^2z = 0, \qquad (5.1)$$

and WU and UV have equations that may be written down from equation (5.1) by cyclic change of x, y, z and a, b, c.

YZ has equation (3.1) and ZX and XY have equations (3.2) and (3.3).

Sides VW and YZ meet at the point L, which therefore has co-ordinates $L(0, -b^2, c^2)$. Similarly M and N have co-ordinates $M(a^2, 0, -c^2)$ and $N(-a^2, b^2, 0)$. It follows that L, M, N are collinear and line LMN has equation

$$x/a^2 + y/b^2 + z/c^2 = 0. \qquad (5.2)$$

It is fairly easy to show that LMN is the polar of K with respect to Σ, and also the polar of K with respect to the conic UVWXYZ, and is perpendicular to the Brocard axis.

6. The sides of DEF

The equation of EF is

$$x/a^2 = 2y/b^2 + 2z/c^2 \qquad (6.1)$$

The equations of FD and DE may now be written down from equation (6.1) by cyclic change of x, y, z and a, b, c.

Note that the sides BC and EF pass through L. Similarly CA and FD pass through M and AB and DE pass through N.

7. The points L_2, M_2, N_1, L_1, N_2, M_1 and the conic passing through these points

The point L_2 = XY^WU and has co-ordinates $(a^2, 3b^2, -c^2)$.
The point M_2 = XY^VW and has co-ordinates $(3a^2, b^2, -c^2)$.
The point N_1 = ZX^VW and has co-ordinates $(3a^2, -b^2, c^2)$.
The point L_1 = ZX^UV and has co-ordinates $(a^2, -b^2, 3c^2)$.
The point N_2 = YZ^WU and has co-ordinates $(-a^2, 3b^2, c^2)$.
The point M_1 = YZ^UV and has co-ordinates $(-a^2, b^2, 3c^2)$.

The conic through these six points has an equation of the form (3.4) with
$u = 2b^4c^4$, $v = 2c^4a^4$, $w = 2a^4b^4$, $f = 11a^4b^2c^2$, $g = 11a^2b^4c^2$, $h = 11a^2b^2c^4$.

Using equations (3.6) the centre S of this conic has co-ordinates (x, y, z), where

$$\begin{aligned} x &= a^2(11(b^2 + c^2) - 13a^2), \\ y &= b^2(11(c^2 + a^2) - 13b^2), \\ z &= c^2(11(a^2 + b^2) - 13c^2). \end{aligned} \tag{7.1}$$

It may now be checked that S lies on the Brocard axis and that LMN is the polar of K with respect to this conic also.

8. The points A', B', C', D', E', F' and the conic passing through these points

The equations of the sides of triangle DEF are given in Section 6.

The equations of AD, BE, CF are $c^2y = b^2z$, $a^2z = c^2x$, $b^2x = a^2y$ respectively.

The point A' = EF^AD and has co-ordinates $(4a^2, b^2, c^2)$.
The point B' = FD^BE and has co-ordinates $(a^2, 4b^2, c^2)$.
The point C' = DE^CF and has co-ordinates $(a^2, b^2, 4c^2)$.
The point D' = BC^AD and has co-ordinates $(0, b^2, c^2)$.
The point E' = CA^BE and has co-ordinates $(a^2, 0, c^2)$.
The point F' = AB^CF and has co-ordinates $(a^2, b^2, 0)$.

It may now be shown that these six points lie on a conic, when expressed in the form (3.4) has

$$u = b^4c^4,\ v = c^4a^4,\ w = a^4b^4,\ f = -a^4b^2c^2,\ g = -a^2b^4c^2,\ h = -a^2b^2c^4. \tag{8.1}$$

This conic is the Brocard inellipse and it has centre R with co-ordinates

$$(a^2(b^2 + c^2), b^2(c^2 + a^2), c^2(a^2 + b^2)). \tag{8.2}$$

It may now be checked that R lies on the Brocard axis and that the polar of K with respect to this inellipse is the line LMN.

9. Review of points on the line LMN

The following lines pass through L: BC, VW, YZ, EF, L_1L_2, B'C, E'F'. The first four of these have already been mentioned. The equations of the last three are:

L_1L_2	$2x/a^2 = y/b^2 + z/c^2$,	(9.1)
B'C'	$5x/a^2 = y/b^2 + z/c^2$,	(9.2)
E'F'	$x/a^2 = y/b^2 + z/c^2$.	(9.3)

It may be checked that L $(0, -b^2, c^2)$ lies on each of these lines.

Seven similar lines pass through M and N.

Points on LMN such as Q $(a^2, -2b^2, c^2)$ have the following lines passing through it: AF, CD, BE (the same as VY), UZ, WX, L_2N_1, N_2L_1 showing it to be a vertex of a reverse perspective of triangles ABC, DEF and also UVW, XYZ.

10. The porisms

Triangles ABC and DEF are inscribed in the circumcircle Σ and have sides that touch the Brocard inellipse at points A', B', C', D', E', F'. Thus a porism is established of triangles inscribed in Σ and circumscribing the Brocard inellipse.

Triangles UVW and XYZ are inscribed in the conic centre T (see Section 3) and circumscribe (by construction) the circumcircle Σ, touching it at A, B, C, D, E, F. Thus a second porism is established.

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