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Properties of a Particular Tucker Circle

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Abstract: Three circles determine a particular Tucker circle and their centres exhibit some interesting properties.

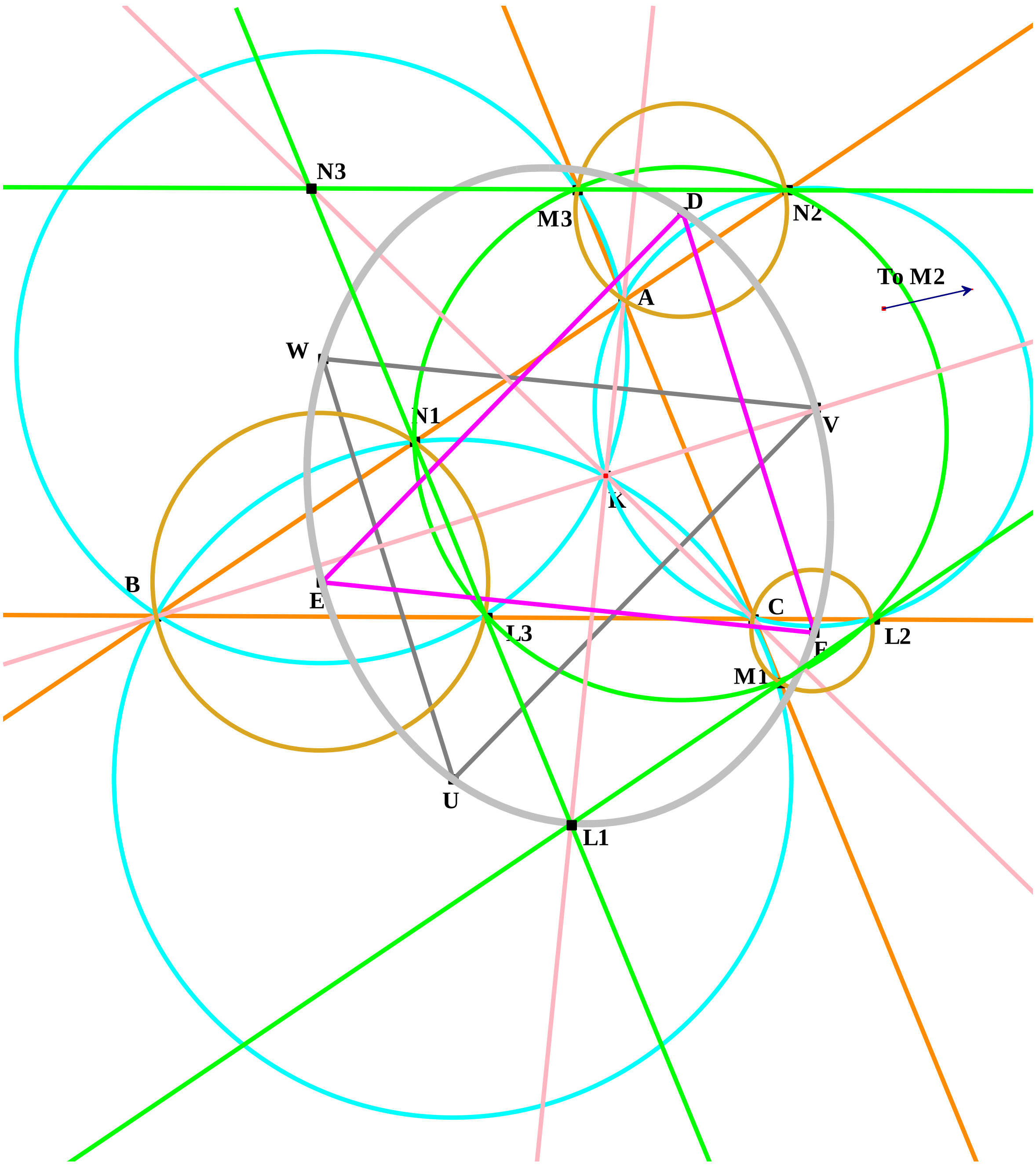

Fig. 1
A Tucker circle and its centres

1. Introduction

Given a triangle ABC and its symmedian point K a construction of Fig. 1 starts by drawing the circles BKC, CKA, AKB. Circle BKC meets AB at N_1 and AC at M_1. Points L_2 and N_2 on CA are similarly defined by circle CKA and points M_3 and L_3 are similarly defined by circle AKB.

The lines L_2M_1 and N_1L_3 meet at L_1, lines M_3N_2 and L_2N_1 meet at M_2 and ;ines N_1L_3 and M_3N_2 meet at N_3. We prove that triangle $L_1M_2N_3$ is an enlargement by a factor of two and a rotation of 180^o of triangle ABC about K.

The centres of circles BKC, CKA, AKB are denoted by U, V, W and the centres of circles AM_3N_2 , BN_1L_3, CL_2M_1 are denoted by D, E, F. We prove that a conic passes through the six points U, V, W, D, E, F.

We use areal co-ordinates throughout with ABC as triangle of reference.

2. Circles BKC, CKA, AKB and their intersections with the sides of ABC

All circles have an equation of the form

$$a^2yz + b^2zx + c^2xy + (ux + vy + wz)(x + y + z) = 0, \qquad (2.1)$$

where u, v, w are determined by the three points defining the circle.

The circle BKC has $u = -3b^2c^2/(a^2 + b^2 + c^2)$, v = 0, w = 0 and hence has an equation

$$3b^2c^2x^2 - a^2(a^2 + b^2 + c^2)yz - b^2(a^2 + b^2 - 2c^2)zx - c^2(c^2 + a^2 - 2b^2)xy = 0. \qquad (2.2)$$

It meets CA at M_1 with co-ordinates $(a^2 + b^2 - 2c^2, 0, 3c^2)$ and it meets AB at N_1 with co-ordinates $(c^2 + a^2 - 2b^2, 3b^2, 0)$.

The equations of circles CKA and AKB have equations that may be determined from Equation (2.2) by cyclic change of x, y, z and a, b, c. Point N_2 has co-ordinates $(3a^2, b^2 + c^2 - 2a^2, 0)$ and point L_2 has co-ordinates $(0, a^2 + b^2 - 2c^2, 3c^2)$. Point L_3 has co-ordinates $(0, 3b^2, c^2 + a^2 - 2b^2)$ and point M_3 has co-ordinaters $(3a^2, 0, b^2 + c^2 - 2a^2)$.

3. The Tucker circle and the triangle $L_1M_2N_3$

The Tucker circle passes through the six points defined in Section 2 and has an Equation of the form (2.1) with

$$u = 3b^2c^2(2a^2 - b^2 - c^{2)}/(a^2 + b^2 + c^2)^2, \qquad (3.1)$$

$$v = 3c^2a^2(2b^2 - c^2 - a^2)/(a^2 + b^2 + c^2)^2, \qquad (3.2)$$

$$w = 3a^2b^2(2c^2 - a^2 - b^2)/(a^2 + b^2 + c^2). \qquad (3.3)$$

The equation of the line M_3N_2 is

$$(2a^2 - b^2 - c^2)x + 3a^2(y + z) = 0. \qquad (3.4)$$

This line is parallel to BC and similarly N_1L_3 is parallel to CA and L_2M_1 ias parallel to AB, verifying that this circle is indeed a Tucker circle.

The equations of N_1L_3 is

$$(2b^2 – c^2 – a^2)y + 3b^2(z + x) = 0 \qquad (3.5)$$

and the equation of L_2M_1 is

$$(2c^2 – a^2 – b^2)z + 3c^2(x + y) = 0. \qquad (3.6)$$

These last two lines meet at the point L_1 with co-ordinates $(a^2 – 2b^2 – 2c^2, 3b^2, 3c^2)$. Similarly M_2 has co-ordinates $(3a^2, b^2 – 2c^2 – 2a^2, 3c^2)$ and N_3 has co-ordinates $(3a^2, 3b^2, c^2 – 2a^2 – 2b^2)$.

4. Circles AM_3N_2, BN_1L_3, CL_2M_1 and the circle centres U, V, W, D, E, F

The circle AM_3N_2 has equation in the form (2.1) with

$$u = 0, \qquad (4.1)$$
$$v = – 3c^2a^2/(a^2 + b^2 + c^2), \qquad (4.2)$$
$$w = – 3a^2b^2/(a^2 + b^2 + c^2). \qquad (4.3)$$

The equations of circles BN_1L_3 and $CL_2 M_1$ may be obtained from that of circle AM_3N_2 by cyclic change of x, y, z and u, v, w and a, b, c.

The co-ordinates of the centre of a circle with equation of the form (2.1) are given by

$$x = a^4 – a^2(b^2 + c^2 + 2u – v – w) + (b^2 – c^2)(v – w), \qquad (4.4)$$
$$y = b^4 – b^2(c^2 + a^2 + 2v – w – u) + (c^2 – a^2)(w – u), \qquad (4.5)$$
$$z = c^4 – c^2(a^2 + b^2 + 2w – u – v) + (a^2 – b^2)(u – v). \qquad (4.6)$$

Since we have the values of u, v, w for each of the six circles we are in a position to write down the co-ordinates of U, V, W, D, E, F. For U these are

$$x = a^2(a^4 - b^4 + 4b^2c^2 – c^4), \qquad (4.7)$$
$$y = – b^2(a^4 + 5a^2c^2 – b^4 + 3b^2c^2 – 2c^4), \qquad (4.8)$$
$$z = – c^2(a^4 + 5a^2b^2 – 2b^4 + 3b^2c^2 – c^4). \qquad (4.9)$$

The co-ordinates of V and W may be written down from Equations (4.7) – (4.9) by cyclic change of x, y, z abd a, b, c.

The co-ordinates of D are

$$x = a^2(a^4 – 3a^2(b^2 + c^2) + 2(b^4 – 4b^2c^2 + c^4)), \qquad (4.10)$$
$$y = b^2(a^2 – b^2 + c^2)(2a^2 – b^2 – c^2), \qquad (4.11)$$
$$z = c^2(a^2 + b^2 – c^2)(2a^2 – b^2 – c^2). \qquad (4.12)$$

The co-ordinates of E and F may be written down from Equations (4.10) – (4.12) by cyclic change of x, y, z and a, b, c.

5. The conic through the six circle centres U, V, W, D, E, F

This has the form

$$ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad (5.1)$$

where

$$u = -2b^2c^2(2a^2 - b^2 - c^2)(a^4(b^2 + c^2) + 2a^2(b^4 + 3b^2c^2 + c^4) + b^6 - b^4c^2 - b^2c^4 + c^6) \qquad (5.2)$$

and v, w follow by cyclic change of a, b, c.

Also

$$f = -a^2b^2c^2(12a^6 + 7a^4(b^2 + c^2) - 4a^2(b^4 - 7b^2c^2 + c^4) + b^6 + 3b^4c^2 + 3b^2c^4 + c^6) \qquad (5.3)$$

and g, h follow by cyclic change of a, b, c.

6. The relationship between triangles ABC and $L_1M_2N_3$

We find the displacement AK to be $(1/(a^2 + b^2 + c^2))(-(b^2 + c^2), b^2, c^2)$ and $KL_1 = 2AK$. Similarly $KM_2 = 2BK$ and $KN_3 = 2CK$. Thus triangle $L_1M_2N_3$ is an enlargement by a factor of two and a rotation of 180^o of triangle ABC about K.

We do not supply the details but it may be shown that triangle DEF and UVW are congruent and that they are a 180^o rotation of each other about a point J, where KJ = (1/4)KO, O being the circumcentre of ABC. Thus J lies on the Brocard axis (at a point that I confess I have not encountered in any of my previous configurations. See Fig. 2.

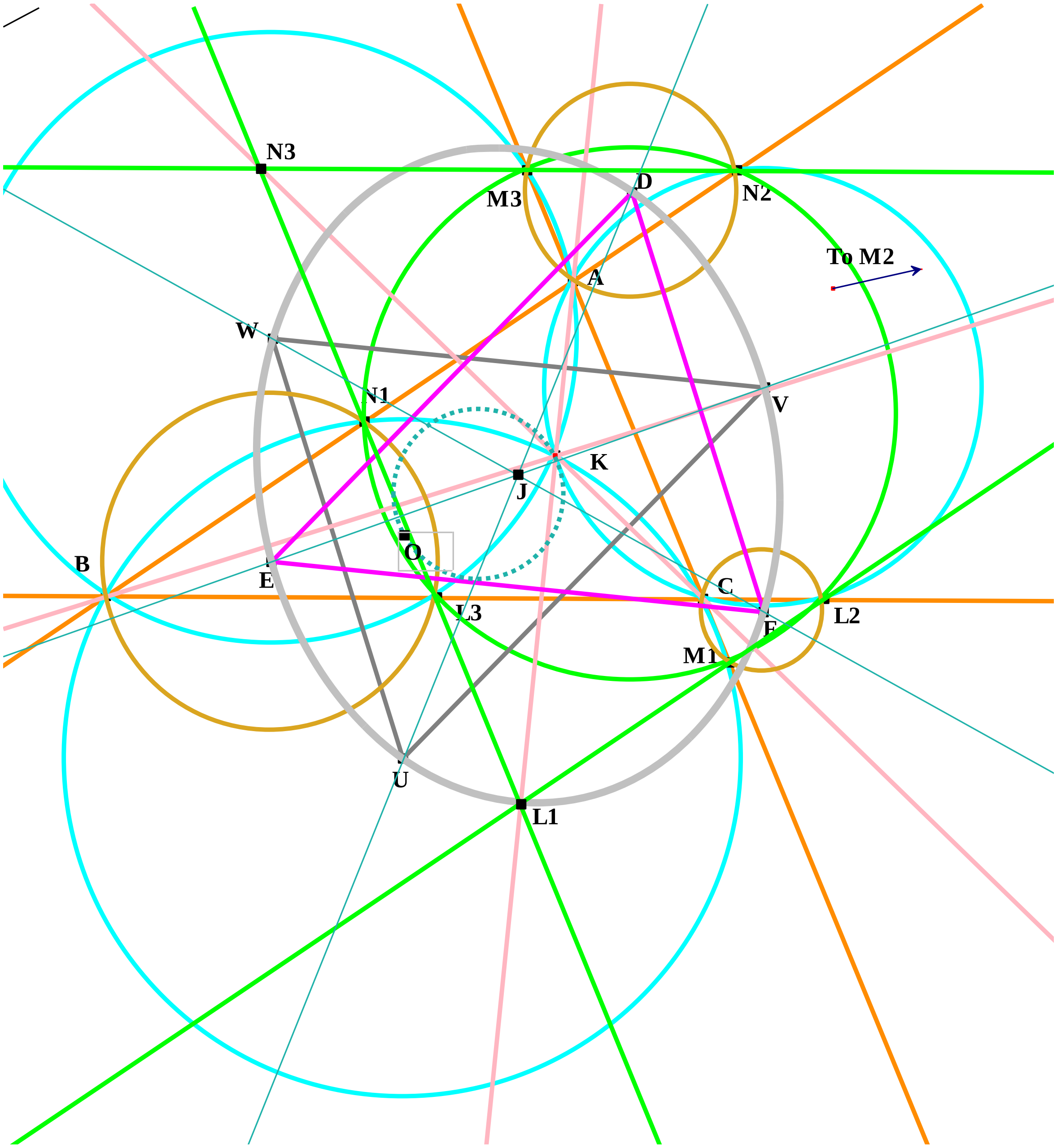

Fig. 2
The point J on the Brocard axis

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