Article: CJB/109/2010

Four Perspectives given a Triangle and its Circumcircle

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Abstract: Given a triangle, its circumcircle and a point not on the circumcircle, it is shown how by varying the position of the point along a line four perspectives may be created.

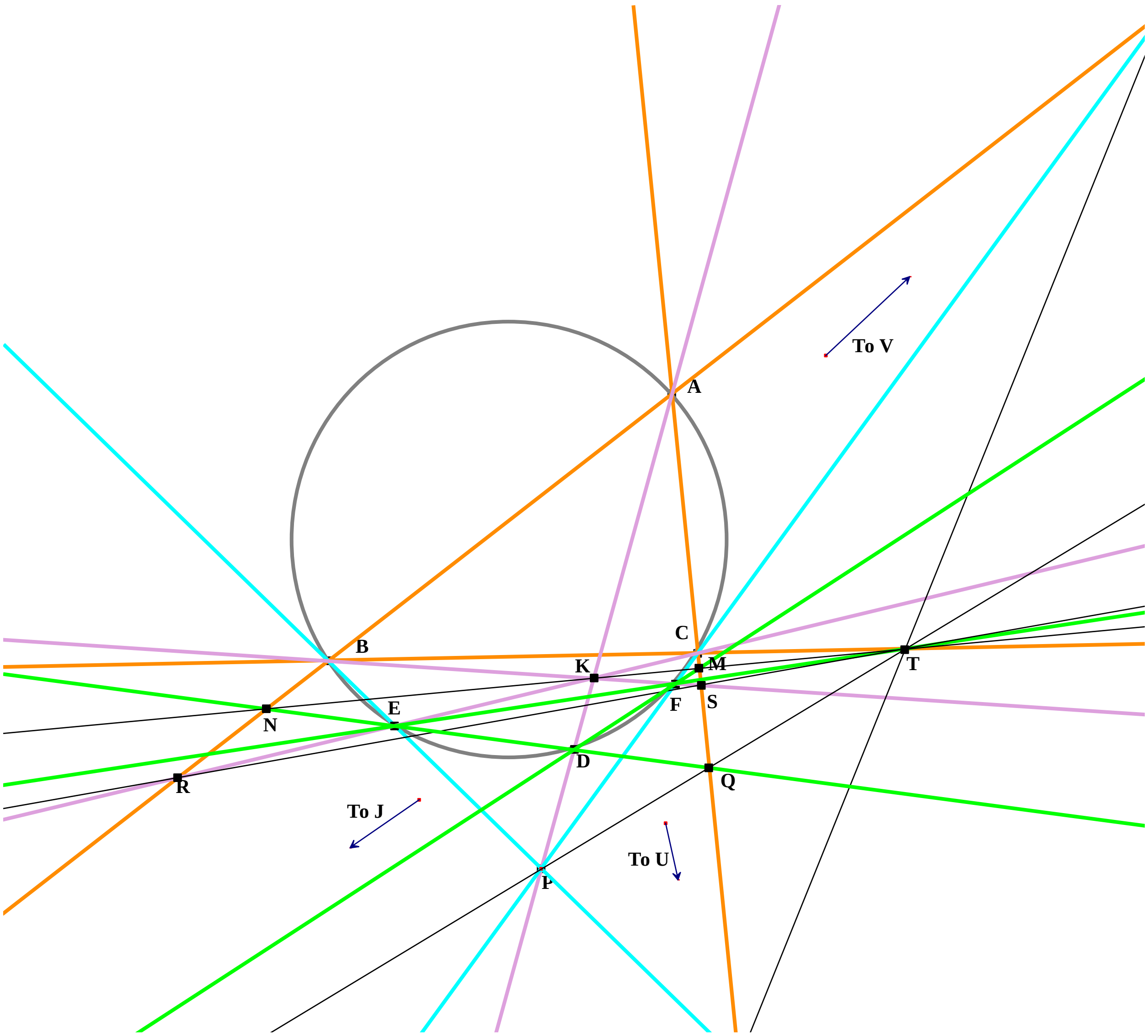

Fig. 1

Triangle ABC and a selected point P produce four perspectives

1. Introduction

Given triangle ABC and its circumcircle S a point P not on the circumcircle may be adjusted so that AKP is a straight line, where K = BF^CE and E = BP^S and F = CP^S. We show in subsequent sections, using Cartesian co-ordinates, that the condition for this to happen is that P should lie on a line determined by the positions of A, B and C. When P is chosen on this line then four perspectives are created (i) ABC and KEF with vertex P, (ii) ABC and PFE with vertex K, (iii) ABC and DEF with vertex P and (iv) ABC and DFE with vertex K. The four axes of perspective have a common point T = EF^BC and K is the harmonic conjugate of P with respect to A and D = AP^S.

2. **Lines AP, BP, CP and points D, E, F**

Let P have co-ordinates (d, e) with $d^2 + e^2 \neq 1$. Later it is shown that for K to be on AP it is necessary that P should lie on a line determined by A, B, C only. The co-ordinates of points A, B, C are $A((1 - a^2)/(1 + a^2), 2a/(1 + a^2))$, with B, C similar but with parameters b, c rather than a.

The equation of the line AP may now be obtained and is

$$((1 + a^2)e - 2a)x - ((1 + a^2)d - (1 - a^2))y - (1 - a^2)e + 2ad = 0. \tag{2.1}$$

The equations of BP, CP may be obtained from Equation (2.1) by replacing a with b and c respectively. The point D where AP meets the circumcircle S has co-ordinates (x, y), where

$$x = (a^2((d + 1)^2 - e^2) - 4ade + e^2 - (d - 1)^2)/(a^2((d + 1)^2 + e^2) - 4be + (d - 1)^2 + e^2), \tag{2.2}$$

$$y = (2(a^2(1 + d)e + a((d^2 - 1) - e^2) - e(d - 1)))/(a^2((d + 1)^2 + e^2) - 4be + (d - 1)^2 + e^2). \tag{2.3}$$

Points E and F have similar co-ordinates but with b, c respectively replacing a.

3. **Lines CE, BF and the point K**

The equations of the lines CE and BF may now be obtained and are

$$\text{CE:}\quad (b(ce - d - 1) + c(d - 1) + e)x - (b(c(d + 1) + e) - ce + d - 1)y + b(ce + d + 1) + c(d - 1) - e = 0. \tag{3.1}$$

$$\text{BF:}\quad (b(ce + d - 1) - c(d + 1) + e)x - (b(c(d + 1) - e) + ce + d - 1)y + b(ce + d - 1) + c(d + 1) - e = 0. \tag{3.2}$$

These lines meet at the point K with co-ordinates (x, y), where

$$x = (b(c(d - e^2 + 1) - de) - cde + d + e^2 - 1)/(b(c(d^2 + d + e^2) - e) - ce + d^2 - d + e^2), \tag{3.3}$$

$$y = (b(d + 1)(ce + d - 1) + c(d + 1)(d - 1) - e(d - 1))/(b(c(d^2 + d + e^2) - e) - ce + d^2 - d + e^2). \tag{3.4}$$

4. **The condition that K lies on AP**

Substitution of the co-ordinates of K in Equations (3.3) and (3.4) into the equation of the line AP in Equation (2.1) provides the condition that K lies on AP and this surprisingly is a linear relation between d and e, showing that the locus of the position of P that implies K lies on AP is a line with equation

$$2(a^2 - bc)e - (a^2(b + c) - 2a(1 + bc) + b + c)d = (a^2(b + c) + 2a(1 - bc) - b - c). \quad (4.1)$$

David Monk (private communication) has pointed out that this line is the line through A passing through the symmedian point.

5. **The four perspectives**

An immediate consequence of K lying on AP is that K and P are then harmonic conjugates of A and D. This is because (AKDP) = C(AEDF) = B(AEDF) = (APDK).

It is also clear that when K lies on ADP then four perspectives are created. These are

(i) Triangles ABC and KEF with vertex P and axis TSR;
(ii) Triangles ABC and PFE with vertex K and axis TUV;
(iii) Triangles ABC and DEF with vertex P and axis TMN;
(iv) Triangles ABC and DFE with vertex K and axis TQJ.

See Fig. 1 where the positions of points T, S, U, V, M, N, Q, J lie. Note that T lies on all four axes.

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