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The Altitudes Create Four Circles, Four Conics and a Polar line

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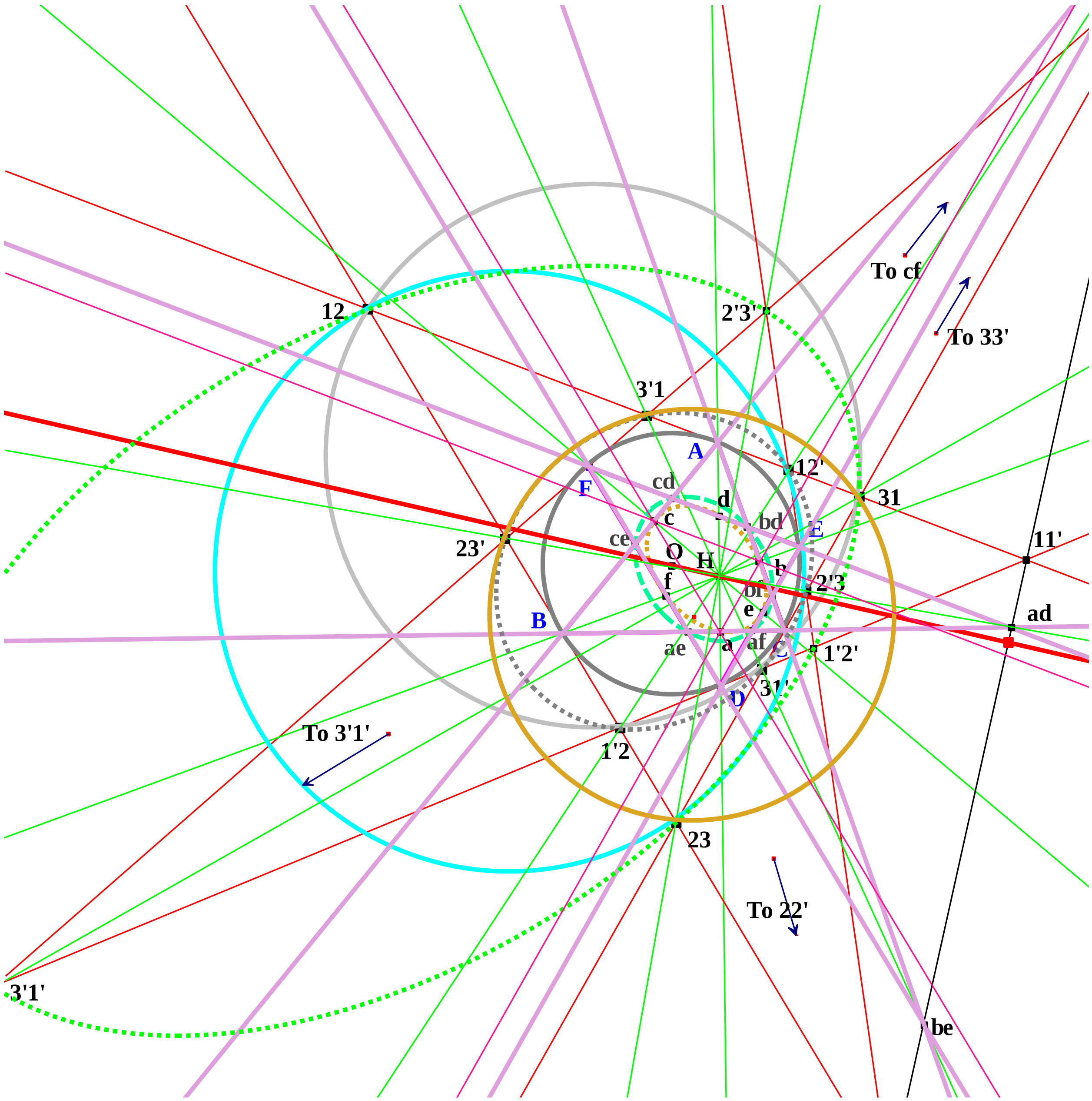

Fig. 1
The four circles, four conics and the polar line of H

Abstract: If the altitudes of a triangle ABC meet the circumcircle at D, E, F, then the six interior intersections of the two triangles and the tangents at the above six points produce a

configuration in which four circles and four conics feature. An analysis is given in which the polar of the orthocentre plays a major role.

1. Introduction

The best introduction is a description of how Fig. 1 was constructed, introducing the conics and circles as they emerge. In the following sections analytic verifications are made of the results, using areal co-ordinates with ABC as triangle of reference. It is true, of course, that many of the results follow immediately from simple synthetic arguments, but the equations of the conics and circles are significant in their own right.

The start, of course, is a triangle ABC and its circumcircle Σ. The altitudes are then drawn meeting Σ at D, E, F. We now have two triangles ABC and DEF. Points are now constructed from the following lines: the sides AB, BC, CA, DE, EF, FD and the tangents to Σ at A. B, C, D,. E, F. The internal points of intersection of the two triangles are cd = AB^EF, ce = AB^FD, ae = BC^FD, af = BC^DE, bf = CA^DE, bd = CA^EF. The external points of intersection are ad = BC^EF, be = CA^FD, cf = AB^DE. It is proved that cd, ce, ae, af, be, cf lie on a conic

Points on the tangents at A, B, C carry labels 1, 2, 3 respectively and tangents at D, E, F carry labels 1', 2', 3'. So 11' is the intersection of the tangents at A and D, 22' is the intersection of the tangents at B and E, 33' is the intersection of the tangents at C and F. It is then the case that 11', 22', 33', ad, be, cf lie on a line, which we prove to be the polar of the orthocentre H with respect to Σ.

If one now thinks of the four points ABDC in that order, we may consider AD and BC to be two perpendicular diagonals of a cyclic quadrilateral. It follows then from a well-known theorem that the tangents at these four points meet in pairs in another circle. It follows that 12, 31, 1'2 and 31' are concyclic. Similarly 23, 12, 2'3, 12' are concyclic as are 31, 23, 3'1, 23'.

The ex-symmedian points of triangles ABC and DEF, namely 12, 23, 31, 1'2', 2'3', 3'1' are proved to lie on a conic. The points 12', 23', 31', 1'2, 2'3, 3'1 are also proved to lie on a conic.

If the altitude AD meets EF at d and BC at a, with e, b, f, c similarly defined, then finally it is shown that these six points lie on a conic.

2. Points D, E, F, the sides EF, FD, DE and the tangents to Σ at A, B, C, D, E, F

We take the orthocentre H of triangle ABC to have co-ordinates (u, v, w), so that the equation of the circumcircle Σ is

$$u(v + w)yz + v(w + u)zx + w(u + v)xy = 0. \qquad (2.1)$$

The equation of AH is wy = vz and this line meets Σ at the point D(– u(v + w), v(2u + v + w), w(2u + v + w)). Similarly E has co-ordinates E(u(2v + w + u), – v(w + u), w(2v + w + u)) and F has co-ordinates F(u(2w + u + v), v(2w + u + v), – w(u + v)).

We are now able to find the equations of the sides of triangle DEF. The equation of EF is

$$vw(v + w)x - wu(u + 2v + w)y - uv(u + v + 2w)z = 0. \qquad (2.2)$$

The equations of FD and DE follow from Equation (2.2) by cyclic change of x, y, z and u, v, w.

The tangent at A to Σ has equation

$$w(u + v)y + v(w + u)z = 0. \qquad (2.3)$$

The tangents at B and C to Σ follow from Equation (2.3) by cyclic change of x, y, z and u, v, w.

The tangent at D to Σ has equation

$$vw(2u + v + w)^2x + wu(v + w)(w + u)\, y + uv(v + w)(u + v)z = 0. \qquad (2.4)$$

The tangents at E and F to Σ follow from Equation (2.4) by cyclic change of x, y, z and u, v, w.

3. Points 12, 23, 31, 1'2', 2'3', 3'1' and the conic through these six points

The tangents at A and B to Σ meet at the point 12 whose co-ordinates are (u(v + w), v(w + u), – w(u + v)). Similarly point 23 has co-ordinates (– u(v + w), v(w + u), w(u + v)) and the point 31 has co-ordinates (u(v + w), – v(w + u), w(u + v)).

Tangents at D and E to Σ meet at the point 1'2' whose co-ordinates are (– uv(v + w), – uv(w + u), $w(u^2 + v^2 + w^2 + 3uv + 2vw + 2wu)$). Similarly point 2'3' has co-ordinates ($u(u^2 + v^2 + w^2 + 2uv + 3vw + 2wu)$, – vw(w + u), – vw(u + v)) and 3'1' has co-ordinates (– wu(v + w), $v(u^2 + v^2 + w^2 + 2uv + 2vw + 3wu)$, – wu(u + v)).

We are now in position to work out the equation of the conic through these six points, which are the ex-symmedian points of the two triangles. It is

$$px^2 + qy^2 + rz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad (3.1)$$

where

$$p = 2v^2w^2(u + v)(v + w)(w + u)(2u + v + w),$$
$$q = 2w^2u^2(u + v)(v + w)(w + u)(u + 2v + w),$$
$$r = 2u^2v^2(u + v)(v + w)(w + u)(u + v + 2w),$$
$$f = u^2vw(v + w)k,$$
$$g = v^2wu(w + u)k,$$
$$h = w^2uv(u + v)k, \qquad (3.2)$$

and where

$$k = 2(u^3 + v^3 + w^3) + 5(uv^2 + uw^2 + vw^2 + vu^2 + wu^2 + wv^2) + 12uvw. \quad (3.3)$$

4. The points 11', 22', 33' and the line joining them

Tangents at A and D to Σ meet at the point 11' whose co-ordinates are

$$(u(v + w)(w – v), – v(w + u)(2u + v + w), w(u + v)(2u + v + w)). \quad (4.1)$$

Points 22' and 33' have co-ordinates that may be written down from (4.1) by cyclic change of x, y, z and u, v, w.

These three points lie on the line with equation

$$vw(2u + v + w)x + wu(2v + w + u)y + uv(2w + u + v)z = 0. \quad (4.2)$$

It may be checked that this is the polar of the orthocentre H with respect to the circumcircle Σ.

5. The points a, b, c, d, e, f and the conic through these six points

The points a, b, c are the points where the altitudes meet the opposite sides and so have co-ordinates (0, v, w), (u, 0, w), (u, v, 0) respectively.

The line EF has Equation (2.2) and AD has equation wy = vz. These lines meet at the point d with co-ordinates d(u(2u + 3v + 3w), v(v + w), w(v + w)). Similarly the points e and f have co-ordinates e(u(w + u), v(2v + 3w + 3u), w(w + u)) and f(u(u + v), v(u + v), w(2w + 3u + 3v)).

We may now find the equation of the conic through the points a, b, c, d, e, f, which is equation (3.1), where now

$$\begin{aligned} p &= 2v^2w^2(v + w), \\ q &= 2w^2u^2(w + u), \\ r &= 2u^2v^2(u + v), \\ f &= – u^2vw(2u + v + w), \\ g &= – v^2wu(u + 2v + w), \\ h &= – w^2uv(u + v + 2w). \end{aligned} \quad (5.1)$$

6. The points ad, be, cf lie on UVW, the polar of H

The point ad = BC^EF. The line EF has equation (2.2) so ad has co-ordinates (0, – v(u + v + 2w), w(u + 2v + w)). Points be, cf have co-ordinates that may be obtained from those of ad by cyclic change of x, y, z and u, v, w. It may now be checked that these three points all lie on the line UVW with Equation (4.2) and which is the polar of H with respect to Σ.

7. Points 12, 31, 1'2, 31' lie on a circle

Lines AD and BC are perpendicular diagonals of the cyclic quadrilateral ABDC and so the tangents at A, B, D, C form the circle 12, 31, 1'2, 31'.

The tangents at A and B to Σ meet at the point 12 whose co-ordinates are (u(v + w), v(w + u), – w(u + v)). Similarly point 31 has co-ordinates (u(v + w), – v(w + u), w(u + v)).

The tangent at D to Σ has equation

$$(2u + v + w)^2vwx + (v + w)(w + u)wuy + (u + v)(v + w)uvz = 0, \qquad (7.1)$$

and the tangent at B has equation

$$(u + v)wx + (v + w)uz = 0. \qquad (7.2)$$

These two lines meet at the point 1'2 with co-ordinates (– u(v + w), v(3u + 2v + w), w(u + v)). Similarly 31' has co-ordinates (– u(v + w), v(w + u), w(3u + v + 2w)). It may now be checked that the four points lie on the circle with equation

$$vw(u + v)(v + w)(w + u)(u + v + w)x^2 + w^2u(u + v)(v + w)(w + u)y^2 + v^2u(u + v)(v + w)(w + u)z^2 + u(v + w)(((u^2 + u(v + w))(v^2 + 4vw + w^2)) + vw(v^2 + w^2))yz + u^2(v^2 + 6vw + w^2) + u(v + w)(v^2 + 6vw + w^2) + vw(v + w)^2)((u + v)wxy + v(w + u)zx) = 0. \qquad (7.3)$$

In similar manner it may be shown that points 23, 12, 2'3, 12' lie on a circle, as do points 31, 23, 3'1, 23'. So together with the circumcircle we have four circles in the configuration.

8. The points cd, ae, bf, bd, ce, af and the conic through these points

The point cd = AB^EF. The line EF has Equation (2.2) and AB has equation z = 0. The co-ordinates of CD are found to be cd(u(u + 2v + w), v(v + w), 0). Similarly ae has co-ordinates ae(0, v(u + v + 2w), w(w + u)) and bf has co-ordinates (u(u + v), 0, w(2u + v + w)).

Now bd = CA^EF and the co-ordinates of bd are found to be bd(u(u + v + 2w), 0, w(v + w)). Similarly ce has co-ordinates ce(u(w + u), v(2u + v + w), 0) and af has co-ordinates af(0, v(u + v), w(u + 2v + w)).

It may now be checked that these six points lie on the conic with Equation (3.1), where now

$$\begin{aligned} p &= v^2w^2(v + w)(2u + v + w), \\ q &= w^2u^2(w + u)(u + 2v + w), \\ r &= u^2v^2(u + v)(u + v + 2w), \qquad (8.1) \\ f &= – u^2vw(u^2 + v^2 + w^2 + 2uv + 3vw + 2wu), \end{aligned}$$

$$g = -v^2wu(u^2 + v^2 + w^2 + 2uv + 2vw + 3wu),$$
$$h = -w^2uv(u^2 + v^2 + w^2 + 3uv + 2vw + 2wu).$$

It may be added that when any pair of triangles with a common circumcircle are in perspective then Cabri indicates that four conics exist of the type that are generated in this paper. The existence of four circles is, however, limited to when the vertex of perspective is the orthocentre of one of the triangles.

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