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Two Triangles, their Ex-Symmedians and four Conics

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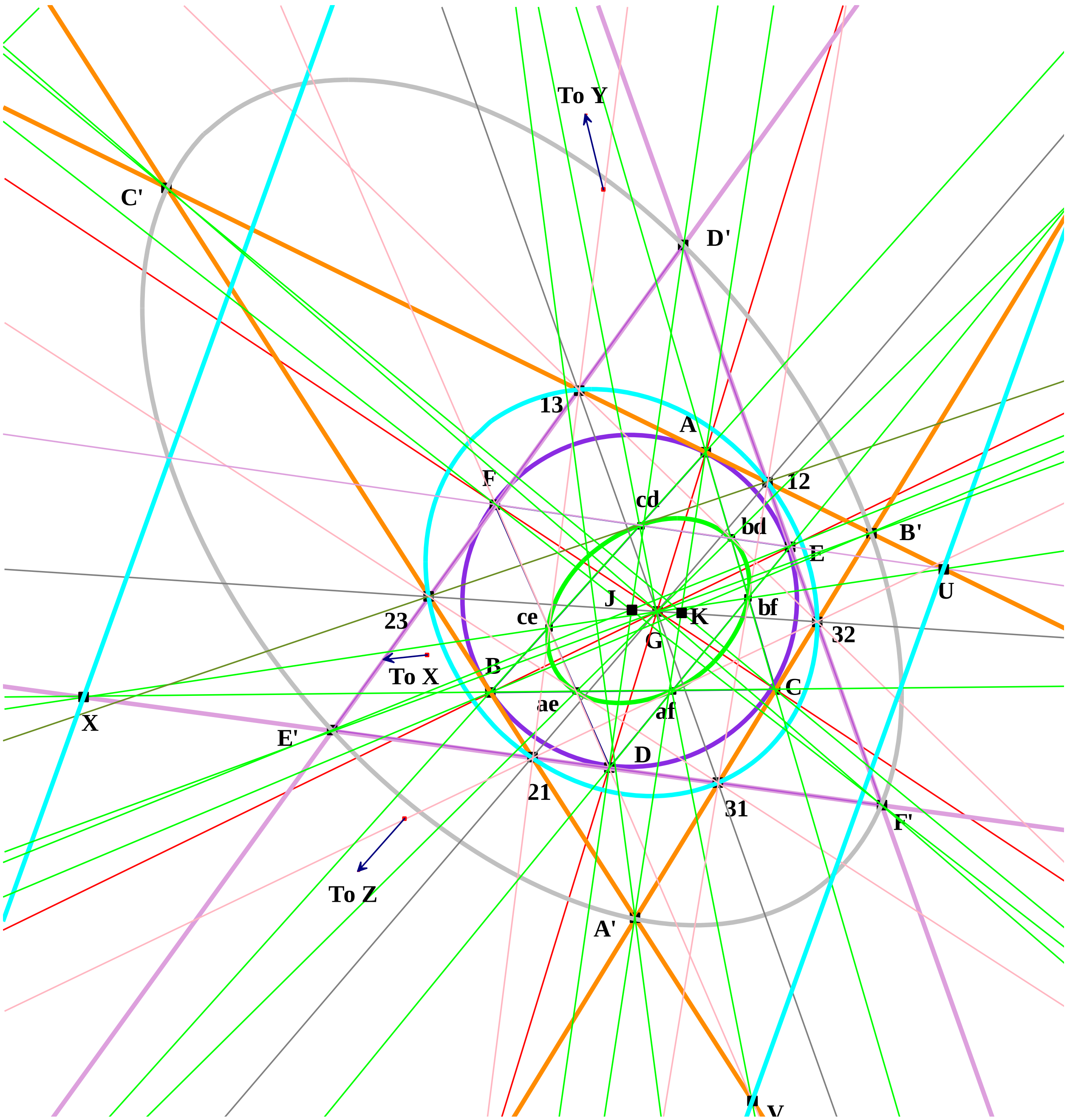

Fig. 1
Two triangles, four conics and a pair of parallel lines

Abstract: The two triangles are in perspective, vertex the centroid of one of them. The six ex-symmedian points lie on a conic, the tangents at the vertex circumscribe the circumcircle and also the intersections of various lines produce six points lying three by three on two parallel lines. Two other conics of interest are also created.

1. Introduction

A triangle ABC, centroid G and circumcircle Σ is given. AG, BG, CG are drawn to meet Σ again at D, E, F respectively. Tangents at A, B, C to Σ are drawn, pairs of which meet at the ex-symmedian points A', B', C'. Similarly tangents to Σ are drawn at D, E, F, pairs of which meet at the ex-symmedian points D', E', F'. Lines B'C' and EF meet at U, lines C'A' and FD meet at V and lines A'B' and DE meet at W. We prove UVW is a straight line. Lines BC and E'F' meet at X, lines CA and F'D' meet at Y and lines AB and D'E' meet at Z. We prove XYZ is a straight line parallel to UVW.

The tangents at A and E meet at the point 12, the tangents at A and F meet at 13, the tangents at B and D meet at the point 21, the tangents at B and F meet at 23, the tangents at C and D meet at 31 and the tangents at C and E meet at 23. We prove that 13, 12, 32, 31, 21, 23 lie on a conic.

The sides of the two triangles ABC and DEF meet at cd = AB^EF, bd = CA^EF, bf = AC^DE, af = BC^DE, ae = BC^FD, ce = AB^FD. We prove that these six points also lie on a conic.

Fig. 1 illustrates all these properties and in addition shows many lines passing through G, for example 13 31. When co-ordinates have been established the equations of such lines may easily be obtained showing that they do in fact pass through G. Such additional properties are left to the reader, as this article is quite long enough as it is!

We use areal co-ordinates throughout with ABC as triangle of reference.

2. Points D, E, F and the ex-symmedians of the two triangles

The line AG has equation y = z and meets the circumcircle Σ with equation

$$a^2yz + b^2zx + c^2xy = 0 \qquad (2.1)$$

at the point D with co-ordinates $D(-a^2, b^2 + c^2, b^2 + c^2)$.

Similarly the co-ordinates of E, F where BG, CG meet Σ are $E(c^2 + a^2, -b^2, c^2 + a^2)$, $F(a^2 + b^2, a^2 + b^2, -c^2)$.

The ex-symmedians A', B', C' of triangle ABC are well-known to have co-ordinates A'($-a^2$, b^2, c^2), B'(a^2, $-b^2$, c^2), C'(a^2, b^2, $-c^2$).

The tangent to Σ at D has equation

$$(b^2 + c^2)^2x + a^2b^2y + a^2c^2z = 0. \qquad (2.2)$$

The equations of the tangents to Σ at E, F may be written down from Equation (2.2) by cyclic change of x, y, z and a, b, c. The ex-symmedian point D' of triangle DEF is the intersection of the tangents at E and F and therefore has co-ordinates (x, y, z), where

$$\begin{aligned} x &= 2b^2c^2 + a^2(a^2 + b^2 + c^2), \\ y &= -b^2(a^2 + b^2 - c^2), \\ z &= -c^2(c^2 + a^2 - b^2). \end{aligned} \qquad (2.3)$$

The co-ordinates of E' and F' may now be written down from those of D' by cyclic change of x, y, z and a, b, c.

3. The conic and lines through the ex-symmedian points

The equation of this conic may now be worked out and is

$$a^2b^2c^2((b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2) + (a^4(b^2 + c^2) + b^4(c^2 + a^2) + c^4(a^2 + b^2))(a^2yz + b^2zx + c^2xy) = 0. \qquad (3.1)$$

It is well-known that AA', BB', CC' meet at the symmedian point K of triangle ABC. The equation of the line DD' is

$$(b^4 - c^4)x + b^2(2c^2 + a^2)y - c^2(a^2 + 2b^2)z = 0. \qquad (3.2)$$

The equations of lines EE', FF' may be written down from Equation (3.2) by cyclic change of x, y, z and a, b, c. These three lines concur at the symmedian point J of triangle DEF with co-ordinates J($2b^2c^2 + a^2(b^2 + c^2 - a^2)$, $2c^2a^2 + b^2(c^2 + a^2 - b^2)$, $2a^2b^2 + c^2(a^2 + b^2 - c^2)$), a point lying on the line through the circumcentre O and the isotomic conjugate of K.

4. The parallel lines UVW and XYZ

The equation of EF is

$$-a^2x + (c^2 + a^2)y + (a^2 + b^2z) = 0, \qquad (4.1)$$

and the equations of FD, DE follow by cyclic change of x, y, z and a, b, c.

The equation of B'C' is

$$c^2y + b^2z = 0, \qquad (4.2)$$

and the equations of C'A' and A'B' follow by cyclic change of x, y, z and a, b, c. The point U is the intersection of EF and B'C' and so has co-ordinates U($b^2 - c^2$, b^2, $- c^2$). The point V is the intersection of FD and C'A' and has co-ordinates V($- a^2$, $c^2 - a^2$, c^2). The point W is the intersection of DE and A'B' and has co-ordinates W(a^2, $- b^2$, $a^2 - b^2$).

It may now be checked that U, V, W all lie on the line with equation

$$(b^2 + c^2 - a^2)x + (c^2 + a^2 - b^2)y + (a^2 + b^2 - c^2)z = 0. \qquad (4.3)$$

The line E'F' has equation

$$c^2a^2x + b^2c^2y + (a^2 + b^2)^2z = 0, \qquad (4.4)$$

and this meets BC at the point X with co-ordinates (0, c^2, $- b^2$). Similarly F'D' meets CA at the point Y with co-ordinates ($- c^2$, 0, a^2) and D'E' meets AB at the point Z with co-ordinates (b^2, $- a^2$, 0). Clearly X, Y, Z all lie on the line with equation

$$a^2x + b^2y + c^2z = 0. \qquad (4.5)$$

Note that UVW and XYZ are parallel since they meet at the point with co-ordinates ($b^2 - c^2$, $c^2 - a^2$, $a^2 - b^2$), which lies on the line at infinity with equation $x + y + z = 0$.

Note that the Euler line of triangle ABC is perpendicular to both UVW and XYZ. *Cabri II plus* indicates that the line UVW is the polar of a point on the Euler line OH beyond H.

5. The points 21, 32, 13, 12, 23, 31 and the conic on which they lie

The tangent at D has Equation (2.2) and that at B is $c^2x + a^2z = 0$. These two tangents meet at the point 21 with co-ordinates 21($- a^2$, $b^2 + 2c^2$, c^2). The tangents C and E meet at the point 32 with co-ordinates 32(a^2, $- b^2$, $c^2 + 2a^2$). The tangents at A and F meet at the point 13 with co-ordinates 13($a^2 + 2b^2$, b^2, $- c^2$). The tangents at D and C meet at the point 31 with co-ordinates 31($- a^2$, b^2, $2b^2 + c^2$). The tangents at E and A meet at the point 12 with co-ordinates ($2c^2 + a^2$, $- b^2$, c^2). The tangents at F and B meet at the point 23 with co-ordinates 23(a^2, $2a^2 + b^2$, $- c^2$).

These six points may now be shown to lie on the conic with equation

$$a^2b^2c^2(a^2(b^2 + c^2)x^2 + b^2(c^2 + a^2)y^2 + c^2(a^2 + b^2)z^2)$$
$$+ (a^6(b^2 + c^2) + b^6(c^2 + a^2) + c^6(a^2 + b^2) + 4a^2b^2c^2(a^2 + b^2 + c^2))(a^2yz + b^2zx + c^2xy) = 0. \qquad (5.1)$$

6. The points ae, bf, cd, ce, af, bd and the conic on which they lie

These are the six internal points of intersection of the triangles ABC and DEF. The point ae = FD^BC has co-ordinates (0, $a^2 + b^2$, b^2). The point bf = DE^CA has co-ordinates (c^2, 0, $b^2 + c^2$). The point cd = EF^AB has co-ordinates ($c^2 + a^2$, a^2, 0). The point ce= FD^AB has co-ordinates (b^2, $b^2 + c^2$, 0). The point af = DE^BC has co-ordinates (0, c^2, $c^2 + a^2$). The point bd = EF^CA has co-ordinates $(a^2 + b^2, 0, a^2)$. These six points may now be shown to lie on the conic with equation

$$a^2(b^2 + c^2)x^2 + b^2(c^2 + a^2)y^2 + c^2(a^2 + b^2)z^2 - (2b^2c^2 + a^2(a^2 + b^2 + c^2))yz - (2c^2a^2 + b^2(a^2 + b^2 + c^2))zx - (2a^2b^2 + c^2(a^2 + b^2 + c^2))xy = 0. \quad (6.1)$$

It may be asked which of the conics remains if a perspective point other than G is used to create Triangle DEF. The answer is that *Cabri II plus* indicates all four conics remain. It is, however, beyond my computing power to prove it. G was chosen because the working was manageable.

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