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Circles though a point in an Equilateral Triangle

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Abstract: Given three points D, E, F lying on the medians of an equilateral triangle ABC the conditions are determined for the circles BCD, CAE, ABF to have a common point.

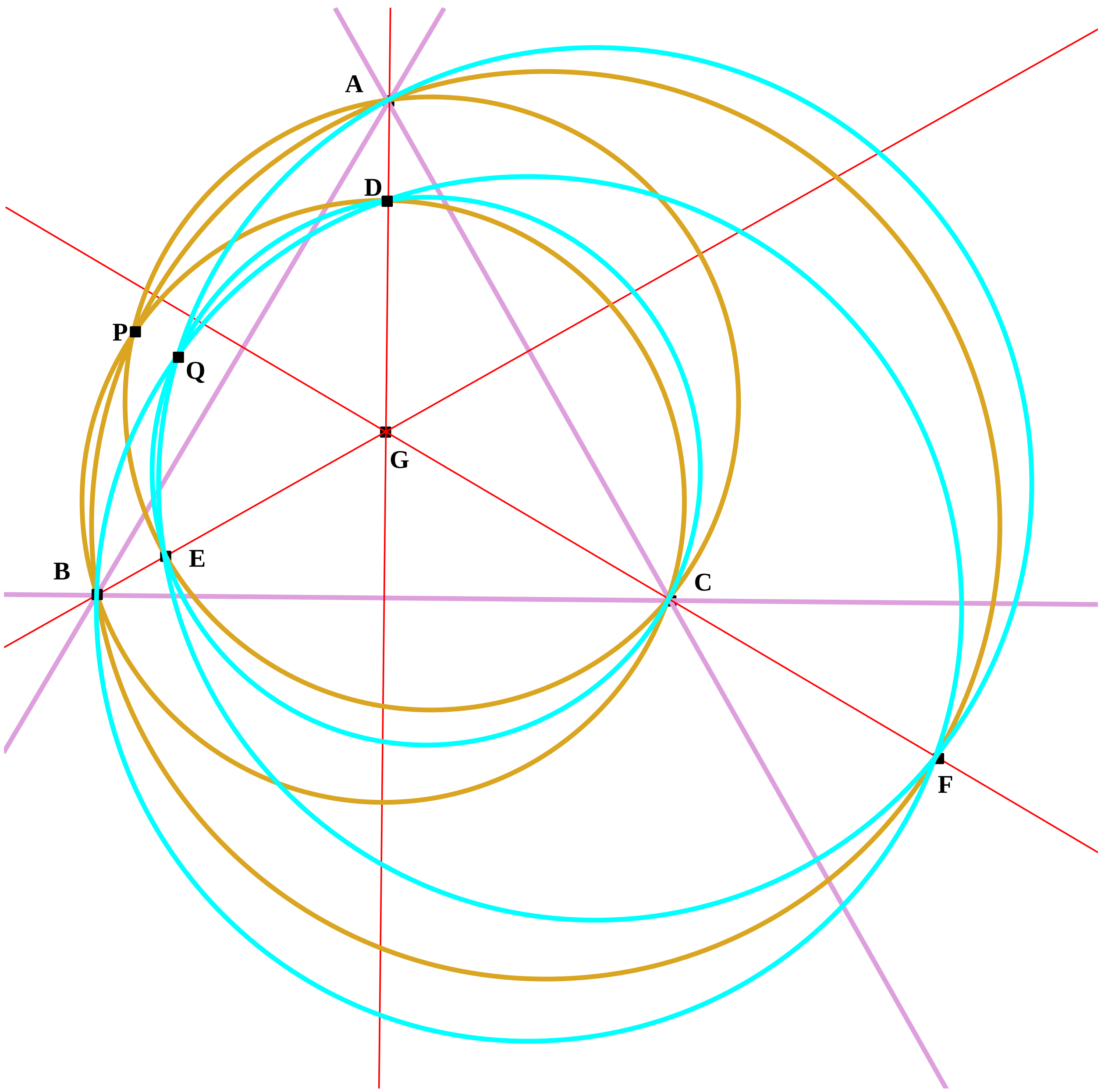

Fig. 1
Showing positions of D, E, F in order that the two sets of circles have a common point

1. Introduction

The purpose of this short article is to show that if E, F are fixed on the medians BG, CG of an equilateral triangle ABC, then there are two positions of D on the median AG for which circles BCD, CAE, ABF have a common point P. It then holds that with D, E, F in either set of positions circles AEF, BFD, CDE also have a common point Q, see [1] and see Fig. 1. *Cabri II plus* indicates that what holds for an equilateral triangle also holds for any triangle and for any set of Cevians (not just the medians). The general cases, however, are technically very involved and presenting them would not really be helpful.

In proving the present propositions areal co-ordinates are used throughout with ABC as triangle of reference. See Bradley [2, 3] for an account of these co-ordinates and how to use them.

2. The points D, E, F and the circles BCD, CAE, ABF

Arbitrary points D, E, F lying on the medians AG, BG, CG of an equilateral triangle ABC and centroid G have co-ordinates $D(1 – 2d, d, d)$, $E(e, 1 – 2e, e)$, $F(f, f, 1 – 2f)$, where d, e, f are some real numbers.

The equation of circle BCD is

$$yz + zx + xy + d(3d – 2)x(x + y + z)/(1 – 2d) = 0. \qquad (2.1)$$

The equations of circles CAE, ABF may now be written down by cyclic change of x, y, z and d, e, f.

3. The common point P of intersection of the three circles

The circles BCD and CAE have common chord

$$(2e – 1)(3d – 2)dx + (e(3e – 2) – 2de(3e – 2))y = 0. \qquad (3.1)$$

The equation of the common chord of circles CAE and ABF may be written down by cyclic change of x, y, z and d, e, f.

These two common chords meet at a point P with co-ordinates (x, y, z), where

$$\begin{aligned} x &= (2d – 1)/\{d(3d – 2)\}, \\ y &= (2e – 1)/\{e(3e – 2)\}, \\ z &= (2f – 1)/\{f(3f – 2)\}. \end{aligned} \qquad (3.2)$$

This point lies on all three circles if either

$$d = (3(e + f) – 3ef – 2)/\{3(e + f – 1)\}, \qquad (3.3)$$

or

$$d = (3ef - e - f)/(6ef - 3e - 3f + 1). \tag{3.4}$$

This shows that if E and F are fixed, then there are two positions of D for which the three circles have a common point.

In Article 19, see [1], we call this situation a case of circular perspective between circles ABC and DEF (the triangles ABC and DEF in this case also being in perspective). And we prove in that article that this relationship is symmetric, and hence circle AEF, BFD, CDE also have a common point Q.

References

1. C. J. Bradley, Article CJB/2010/19 of this series;
2. C.J.Bradley, *Challenges in Geometry*, Oxford (2005);
3. C.J.Bradley, *The Algebra of Geometry*, Highperception, Bath (2007).

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