Asymptotics of cellular buckling close to the Maxwell load

Short title: Asymptotics of cellular buckling

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Abstract

We study the deformation of an elastic strut on a nonlinear Winkler foundation subjected to an axial compressive load P. Using multi-scale analysis and numerical methods we describe the localised, cellular, post-buckled state of the system when P is removed from the critical load $P = 2$. The solutions, and their modulation frequencies, differ significantly from those predicted by weakly nonlinear analysis very close to $P = 2$. In particular, when P approaches the Maxwell load P_M the localised solutions approach a large amplitude heteroclinic connection between an unbuckled solution and a periodic solution. An asymptotic description of P_M in terms of the system parameters is given. The agreement between the numerical calculations and the asymptotic approximations is striking.

Keywords: Buckling; Post-buckling; Maxwell criterion; Strut; Destabilization; Restabilization

1 Introduction

Modern post-buckling analysis of elastic structures is almost always built around the concept of bifurcation from a critical unbuckled state. Following the pioneering work of Koiter [15], linear eigenvalue analysis will define the critical loads and mode shapes of buckling, whereupon asymptotic expansion – based perhaps on perturbation theory – can be used to build up a picture of the large-deflection (post-buckling) response. There is, however, a class of problems for which such information is too localized to be of much practical value. For systems that first destabilize but subsequently restabilize, the buckling profile of the final equilibrium state can appear to be completely unrelated to the transitory unstable states given by linearized theory and subsequent expansion. Indeed, in some circumstances, all buckling information can be lost within a linearization: examples are found in kink-banding [11], in the buckling of railway tracks [14] and pipelines [9, 8], and are also closely mirrored in the buckling responses of some shell structures, notably the axially-compressed cylindrical shell [10].

In such cases the Maxwell stability criterion, where stability is taken to rest only with *global* as opposed to *local* energy minima, has been found a useful conceptual tool [12];

it redefines, for example, the pre-buckled but "super-sensitive" energy minima that can arise in such situations as unstable. Under parametrically-controlled loading or end-displacement, interest then naturally falls on the point where the global energy minimum switches from the pre-buckled to the post-buckled state, shifting the focus from unnaturally high, or infinite, critical loads or displacements to their lower bound *Maxwell* counterparts, where the switch takes place. Observe that if we define the global energy of an unbuckled state as zero, the Maxwell condition occurs when the energy of the buckled state will also be zero. One of the main purposes of this paper is to show how the Maxwell condition arises naturally in an asymptotic analysis, using the method of multiple scales [2] [7], of the fourth order differential equation of a buckled system. The identification and elaboration of the accompanying multiplicity of response, in positions that are remote from the critical bifurcation point, offers new challenges to asymptotic modelling.

This paper offers a new asymptotic treatment for spatial instabilities, based on a general expansion that avoids tying itself too rigidly to the critical bifurcation point as described above. The analysis is applied to a recognized test example [12], that of an elastic strut supported by a nonlinear Winkler foundation that first destiffens and then restiffens under the application of an applied load P.

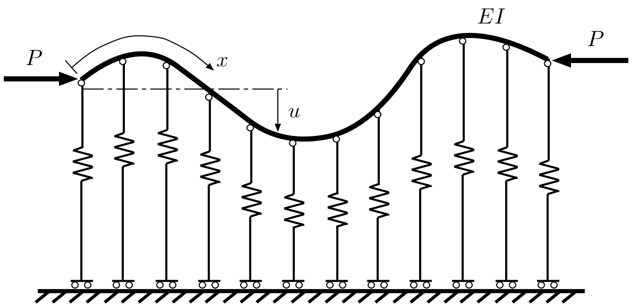

Figure 1.1: *Deformation of a singler layer on a Winkler foundation*

This simple model is often used as a paradigm for studying the behaviour seen in more complex systems such as the axially compressed cylindrical shell. Experimental evidence for this latter problem, reported in [24], indicates clearly the presence of localised and restabilised solutions. Theoretical and numerical studies of the resulting cellular buckling patterns are given in [12].

Since the model we study in this paper is equivalent to the Swift-Hohenberg equation with a general nonlinearity, it is not suprising that the approach has much in common with earlier mathematical formulations [6]. For example, our analysis agrees with the wave number selection for periodic solutions of a symmetric version of the Swift-Hohenberg equation near the classical bifurcation point [3],[17],[20]. As with other cross-fertilizations from nonlinear dynamics to solid mechanics, the analysis has a number of distinctive features. Notably, because spatial differential equations typically are fourth order, new combinations of exponential and harmonic components appear that are not seen in second order equations. Based on scaling arguments, our analysis is capable of moving smoothly from a first form of expansion geared principally for description close to the classical critical point, to a second that focusses principally on larger amplitude heteroclinic responses that are linked to the Maxwell critical load. Non-local asymptotic descriptions are thus developed that avoid pitfalls of too-local analysis, as described above. Comparisons with numerics demonstrate clear superiority over earlier asymptotic approaches to such problems.

The simple model illustrated above may have periodic, quasiperiodic, (spatially) chaotic and localised (homoclinic and heteroclinic) deformations. In this paper we will consider the periodic, homoclinic and heteroclinic solutions. The main results described are (i) an accurate reproduction of the mean form of the bifurcation diagram (excluding exponential perturbations) over a relatively wide range of values of P, (ii) an accurate description of the evolution of homoclinic orbits towards heteroclinic ones as P approaches the Maxwell load P_M. This description is all in terms of a single second order complex amplitude equation. The agreement between the asymptotic results and the numerically computed results is striking, in particular we can estimate P_M to high accuracy. The numerical computations also clearly show the effect of the beyond all orders terms in the asymptotic expansion which also arise in the Swift-Hohenberg equation near the classical bifurcation point at $P = 2$ [3],[17],[20]. A significant feature of these calculations is that they give a good description of the post-buckled state as long as the difference between P and a critical load $P = 2$ (where the unbuckled state first becomes linearly unstable) is smaller than or comparable to a crucial balance parameter ν which depends upon higher order terms in the equation. As $2 - P$ increases the qualitative form of the solutions changes significantly from that found very close to $P = 2$ with a significantly different modulation frequency from that obtained by a linearisation about $P = 2$. In Sections 2–4 we present the asymptotic results for a general nonlinear model. In Sections 5–6 we show how the asymptotic result for the Maxwell load corresponds to $L = 0$ and $H = 0$, for L and H the Lagrangian and Hamiltonian integrals of the systems, respectively. In Section 7 we compare the results to numerical calculations and contrast with previous analyses [22], [12], [3],[17] which are limited to P very close to 2.

2 Problem description and motivation of results

In the following sections we demonstrate the derivation of an amplitude equation for oscillatory solutions (which will include periodic, homoclinic and heteroclinic solutions) of the following partial differential equations

$$\mathcal{L}u + f(u) \equiv u'''' + Pu'' + u + f(u) = 0 \tag{2.1}$$

$$f(u) = a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5 + \dots . \tag{2.2}$$

This equation describes the deviation $u(x)$ of a nondimensionalized strut with a non-zero bending stiffness, which is subjected to an axial compressive load P and which is supported by a nonlinear Winkler foundation of resistance $u + f(u)$. Such equations have been considered for example [5] as models for the compression of a single rock layer surrounded by a resistive medium.

This equation is the first variation of the *Lagrangian integral* L given by

$$L \equiv \int \left[\frac{(u'')^2}{2} - P\frac{(u')^2}{2} + \frac{u^2}{2} + F(u) \right] dx = 0, \quad \text{and} \quad F'(u) = f(u) \tag{2.3}$$

and has as a first integral the *Hamiltonian* H defined by

$$H = u'''u' - \frac{u''^2}{2} + P\frac{u'^2}{2} + \frac{u^2}{2} + F(u). \tag{2.4}$$

In this paper we consider a nonlinear component $f(u)$ that first destiffens and subsequently restiffens. In this case there is a critical load $P = 2$ at which there is a bifurcation from the zero solution into either a periodic or a homoclinic solution. The post-buckling response (for P less than and close to 2) is initially subcritical and unstable, but subsequently restabilizes with movement away from the bifurcation point.

Such an effect can be achieved in a variety of ways and we concentrate here on two of distinction [12]: the general asymmetric foundation,

$$a_2 \neq 0, \quad 0 < a_3 < a_2^2 h(P), \tag{2.5}$$

where $h(P)$ is a smoothly varying positive function of P which is defined later (see 2.17), and the symmetric or reversible foundation,

$$a_2 = a_4 = 0, \quad a_3 < 0, \quad a_5 > 0, \tag{2.6}$$

which has the additional symmetry

$$f(-u) = -f(u).$$

Each of the two inequalities corresponds to a subcritical buckling response. Similar behaviour occurs for both examples in the respective limits of

$$a_3 \to a_2^2 h(P) \qquad \text{and} \qquad a_3/\sqrt{a_5} \to 0,$$

which we shall examine in this paper. The case given by (2.6) is somewhat easier to analyse as it admits symmetric solutions with zero mean and for which $u(x) = -u(T - x)$ for some T. For completeness we will present a general derivation for the harder and more general case (2.5) and compare the asymptotic results with numerics for both cases.

2.1 Overview and description of the amplitude equation

We perform our analysis by deriving a reduced amplitude equation. The amplitude equation gives the evolution of the envelope of a periodically modulated solution, which takes the form

$$u(x) \sim \delta A_1(X)e^{i\omega x} + \text{c.c.} + \delta^2 \left[A_0(X) + A_2(X)e^{2i\omega x} + \text{c.c.} \right] + \mathcal{O}\left(\delta^3\right) ..., \tag{2.7}$$

where X is a slow spatial variable, δ is small in a manner that is identified presently and ω is to be determined. We find that the amplitude equation has both localised (homoclinic) solutions and fixed points corresponding to periodic solutions of (2.1). At a critical load $P = P_M$ it has a heteroclinic solution connecting zero to the periodic solution corresponding to the fixed point. The asymptotic analysis shows that P_M is precisely the Maxwell load at which the periodic solution has Lagrangian $L = 0$ where L is as given in (2.3). When $P < P_M$ there is no homoclinic solution.

The formal derivation of the amplitude equation makes use of two asymptotic limits, the global stabilisation limit of $a_3 \to a_2^2\ h(P)$, or equivalently $a_3/\sqrt{a_5} \to 0$, and the load limit $P \to 2$. The traditional approach of the response of a buckled layer, based on linear analysis, has been to look at the case of $P \to 2$ in isolation and to deduce the existence of homoclinic solutions only. However, now by introducing a *balance parameter* ν proportional to $a_3 - a_2^2 h(P)$, or to $a_3/\sqrt{a_5}$, which is a measure of global stability, we can greatly extend the results. In particular we study the case when $\sqrt{2-P}$ is both small in absolute magnitude and comparable to ν, which leads to the heteroclinic-like behaviour mentioned earlier. Through the appropriate scaling of these parameters the amplitude equation describes both the small amplitude oscillations and the larger amplitude heteroclinics.

Once these scalings are introduced the derivation of the amplitude equation is essentially a standard calculation [16]. We note special features of the underlying equation which are related to the smoothing properties of the linear operator $\mathcal{L}$ and the scalings which are used in the derivation. The linear operator in (2.1) is fourth order in space, while the amplitude equation is second order in a slow spatial variable, and is of a form that has been seen in other contexts [6]. In general the underlying fourth-order equation is nonintegrable, whereas the envelope solution assumes integrability.

To motivate the full derivation of the amplitude equation of Section 3 we will quickly skip forward and have a look at the derived amplitude equation. We discuss two "generic" cases. In the first, a_2 is bounded away from zero, as in (2.5). If we set

$$u(x) = R(x)e^{i\omega x} + \text{c.c.} + \text{ other modes}$$

the resulting amplitude equation in unscaled form is then

$$2PR_{xx} = \varepsilon^2 R - 2\nu R^3 + gR^5 \tag{2.8}$$

with the constant and higher mode terms in $u(x)$ closely coupled to $R(x)$. Later we show that to leading order

$$\omega^2 = P/2, \quad \varepsilon^2 = 1 - \frac{P^2}{4},$$

and g is a constant of $\mathcal{O}(1)$. If P is very close to 2 ($\sqrt{P-2} \ll \nu$), then $h(P) \approx 38/27$ as in [22] and [12] and $\nu \sim 3(38a_2^2/27 - a_3)/2$. Then the cubic terms in this expression dominate, balancing with the linear term, and the quintic terms can be neglected. This occurs when $\varepsilon \ll \nu$, and leads to small amplitude homoclinic solutions of the form

$$R(x) = \frac{\varepsilon}{\nu^{1/2}} \operatorname{sech}\left(\frac{X}{\sqrt{2P}}\right) \quad \text{with} \quad X = \varepsilon x. \tag{2.9}$$

In contrast, if $\nu \sim \varepsilon$ then the quintic and cubic terms also balance, leading to near heteroclinic solutions with amplitude $R = \mathcal{O}(\varepsilon^{1/2})$. In particular, if

$$\varepsilon^2 = \frac{3\nu^2}{4g} \tag{2.10}$$

then there is a heteroclinic connection for which

$$R^2(x) = \frac{\varepsilon}{\gamma\,(1+e^{\theta X})} \quad \text{with} \quad \gamma = \sqrt{\frac{g}{3}} \quad \text{and} \quad \theta = \sqrt{\frac{2}{P}}. \tag{2.11}$$

The equation (2.10) leads to an expression for the critical Maxwell load P_M at which there is a heteroclinic connection linking the unbuckled and buckled periodic states.

The second "generic" case occurs when $a_2 = a_4 = 0$ as in (2.6). Making a preliminary rescaling of the form

$$u \;\to\; a_5^{-1/4} u \tag{2.12}$$

we may then write (2.1) as

$$\mathcal{L}u + \frac{a_3}{\sqrt{a_5}} u^3 + u^5 + \ldots = 0 \tag{2.13}$$

The corresponding amplitude equation then has exactly the same form as (2.8) with a different constant g and with ν now defined by

$$2\nu/3 = -a_3/\sqrt{a_5}.$$

2.2 Scalings

In order to obtain the behaviours outlined above from one amplitude equation it is essential that the correct scalings of u and x are introduced, so that the terms in the equation balance. A hint of the correct form of the scaling is given by the analysis of the amplitude equation described above (2.8), which implies that we should take

$$\delta = \varepsilon/\nu^{1/2} \quad \text{and} \quad X = \varepsilon x$$

in (2.7). This scaling is based on the relative size of the load difference and the balance parameters. We outline the derivation more carefully in this section, concentrating in the main on the general asymmetric foundation (2.5). The entirely parallel analysis for the reversible foundation (2.6) is outlined in Section 4.

We firstly define the *symbol* of ω of the operator $\mathcal{L}$ in (2.1) by

$$\mu(\omega) = \omega^4 - P\omega^2 + 1 = (\omega^2 - P/2)^2 + 1 - P^2/4. \tag{2.14}$$

Now we give a general definition of the size parameters ε and ν which were introduced in the last section. Firstly we set

$$\varepsilon^2 = \mu(\omega) \tag{2.15}$$

as a measure of the distance of P from the critical load at which $P = 2$ and $\omega = 1$. Observe that if $\omega^2 = P/2$ the definition of ε given here and in the previous section are equivalent.

If $a_2 \neq 0$ then we define the balance parameter ν by

$$\nu = a_2^2\left(\mu(2\omega)^{-1} + 2\right) - 3a_3/2 > 0, \tag{2.16}$$

The condition $\nu > 0$ is then equivalent to the inequality $a_3 < a_2^2 h(P)$ so that taking $\omega^2 = P/2$ we have

$$h(P) = 2(\mu(\sqrt{2P})^{-1} + 2)/3. \tag{2.17}$$

When $a_2 = a_4 = 0$, we take

$$\nu \;\; = \;\; -3a_3/(2a_5^{1/2}) > 0 \tag{2.18}$$

The definitions of ν arise from the underlying fourth-order variation of the total potential function. If $\nu > 0$ at $P = 2$, then the equilibrium state is unstable and the bifurcation sub-critical [21]. In contrast, if $\nu < 0$ then the bifurcation is super-critical.

In all of the following analysis we take $\varepsilon^2 \ll 1$, while ν will vary between $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$.

In the expression (2.7) for u we allow δ to vary in size with the parameters ε, ν by defining

$$\delta^2 = \frac{\varepsilon^2}{\nu}. \tag{2.19}$$

We see from (2.8) that this is consistent with taking $A_1 = \mathcal{O}(1)$ in (2.7). In particular, as ν decreases, the amplitude of u increases. As we have already seen in sketch form in Section 2.1, and will confirm below, there are two significant ranges of values of ε and ν

- $\varepsilon \sim \nu$ corresponds to *heteroclinic connections* and larger amplitudes,
- $\varepsilon \ll \nu$ corresponds to smaller amplitudes, and *homoclinic solutions.*

Thus, if ε is fixed, then as ν decreases we find the transition between low amplitude homoclinic solutions and the larger amplitude heteroclinic solutions of the amplitude equation.

Note that in both cases we have assumed that $\varepsilon^2 \ll 1$. As we show below, the modulation frequency is given by $\omega^2 = P/2$ to leading order. Combined with the expression for $\mu(\omega)$ (2.14), $\varepsilon^2 \ll 1$ suggests that P is "near" 2. While this appears to be the same as in the traditional derivation of an amplitude equation based on the bifurcation point $P = 2$, there are significant differences. Traditionally, when the derivation of an amplitude equation is based on the bifurcation analysis alone, a single small parameter is defined in terms of the proximity to the bifurcation point ($\varepsilon^2 = 2 - P$), which is determined from a linear analysis. Then quantities such as the amplitude, bifurcation parameter (load), and frequency are scaled in powers of ε, which then dictates a quadratic relationship between ε and the proximity of the balance parameter ν to a critical value. In Section 8 we show that this quadratic relationship is appropriate only in a very small neighborhood of the bifurcation point.

In contrast, in this paper we define two independent small parameters ε^2 and ν, and the relationship between these small parameters is obtained by balancing linear and nonlinear terms in the amplitude equation in order to describe the transition from homoclinic to heteroclinic connections. In this sense the scalings are based on a nonlinear analysis. This balancing leads to the appropriate scaling of the amplitude as δ. Note that for both $\varepsilon \sim \nu$ and $\varepsilon \ll \nu$ the parameter $\delta \ll 1$, so that the derivation of the amplitude equation in Section 3 is based on a small amplitude expansion together with the balancing of terms in the amplitude equation. Also, the relationship $\omega^2 = P/2$ between the frequency and load is not specified in terms of the bifurcation point, but rather determined as part of the analysis (see Section 2.3). By allowing the relationship of several important parameters to be determined within the context of the derivation of the amplitude, the method of this paper allows greater flexibility in balancing parameters, which allows for results to be valid for a larger range of the load P. We find that this is indeed the case in comparisons with the numerics in Section 7, where we discuss the asymptotic validity in detail.

2.3 The relationship of the frequency to the load

An important feature of our analysis is the identification of a relationship between the load P and the frequency ω of the slowly varying (homoclinic and heteroclinic) solutions. Once the correct scalings are identified, a solvability condition for the fourth order operator in (2.1) gives the leading order relationship,

$$\omega^2 = P/2$$

between the frequency of the oscillations and the load P. At the Maxwell load P_M this is the first term of the asymptotic expression of the condition that $L = 0$ for the periodic solution. In the paper [12] an independent argument based upon the calculus of variations, shows that the condition $L = 0$ is satisfied by that periodic solution which is the limit of an infinite sequence of homoclinic solutions of (2.1) with increasing end-shortening.

In the definitions of ε^2 (2.15) and ν (2.16), we do not specify a value of P and hence do not identify a small parameter in terms of the distance from $P = 2$ and $\omega = 1$, as the linear theory suggests [22]. To make the contrast concrete, the linear theory suggests that near the bifurcation point $P = 2$, appropriate choices of the small parameter ε and ω are

$$\varepsilon^2 = 2 - P \quad \text{and} \quad \omega = 1 - \varepsilon^2/8.$$

In contrast, in our expansion we assume only that $\varepsilon^2 = \mu(\omega)$ is small relative to ν, so that $\varepsilon/\sqrt{\nu}$ is the correct scaling for the amplitude of the first mode. When this is assumed, the condition $\omega^2 = P/2$ follows naturally as a leading order solvability condition for the existence of slowly varying homoclinic solutions. If one defines a small parameter $\hat{\varepsilon}^2 \ll 1$, via $\hat{\varepsilon}^2 = 2 - P$, then the condition $\omega^2 = P/2$ implies $\omega \sim 1 - \hat{\varepsilon}^2/4$, which differs from the linear theory. We will give numerical evidence that the condition $\omega^2 = P/2$ appears to remain valid for the homoclinic solutions, for a very wide range of values of ε and ν.

3 The Amplitude Equation for $\nu = \mathcal{O}(\varepsilon)$ (the Heteroclinic Connection)

Motivated by the previous discussions on scaling, we now derive the amplitude equation, the form of which we have already sketched. In this section we focus on the general non-symmetric case when $a_2 \neq 0$. Essentially the same approach is used in the symmetric case $a_2 = a_4 = 0$, and will be discussed in less detail in Section 4.

Starting with the underlying fourth order equation,

$$\mathcal{L}u + f(u) \equiv u'''' + Pu'' + u + f(u) = 0 \tag{3.1}$$

$$f(u) \quad = \quad a_2u^2 + a_3u^3 + a_4u^4 + a_5u^5 + \ldots, \; a_2 \neq 0 \tag{3.2}$$

we look for an equation for the slowly varying amplitude of the primary mode with frequency ω. The slowly varying spatial variable is $X = \varepsilon x$, as motivated by the scaling introduction (2.11). We then expand the function $u(x, X)$ in terms of the small parameter δ,

$$u(x) \equiv u(x, X) \quad \sim \quad \delta u_1(x, X) + \delta^2 u_2(x, X) + \delta^3 u_3(x, X) + \ldots, \tag{3.3}$$

where $\delta = \varepsilon/\nu^{1/2}$ and ε and ν are given in (2.15)-(2.16). Then we replace $u'(x)$ by the compound expression $u'(x) = u_x + \varepsilon u_X$. For clarity we mainly consider the case

$$\nu = \mathcal{O}(\varepsilon) \Rightarrow \delta = \mathcal{O}(\varepsilon^{1/2}), \tag{3.4}$$

anticipating that we are looking for the heteroclinic solution at the Maxwell load. From the derivatives we then get terms which have coefficients ε^j, which from (3.4) are $\mathcal{O}(\delta^{2j})$. Once these scalings and expansions are chosen, the derivation of the amplitude equation is a straightforward envelope equation calculation.

We now substitute the expansion (3.3) into (2.1), and obtain a sequence of equations for $u_j(x, X)$, $j = 1 - 5$, by equating the coefficients of like powers of δ.

$$\mathcal{O}(\delta): \quad \mathcal{L}u_1 = 0 \tag{3.5}$$

$$\mathcal{O}(\delta^2): \quad \mathcal{L}u_2 = -a_2u_1^2 \tag{3.6}$$

$$\mathcal{O}(\delta^3): \quad \mathcal{L}u_3 = -4\frac{\varepsilon}{\delta^2}(u_1)_{Xxxx} - 2\frac{\varepsilon}{\delta^2}P(u_1)_{Xx} - 2a_2u_1u_2 - a_3u_1^3 \tag{3.7}$$

$$\begin{aligned}\mathcal{O}(\delta^4): \quad \mathcal{L}u_4 = & -a_2u_2^2 - 2a_2u_1u_3 - 3a_3u_1^2u_2 - \frac{\varepsilon}{\delta^2}4(u_2)_{Xxxx} \\ & -2\frac{\varepsilon}{\delta^2}P(u_2)_{Xx} - a_4u_1^4\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{O}(\delta^5): \quad \mathcal{L}u_5 = & -\frac{\varepsilon^2}{\delta^4}P(u_1)_{XX} - \frac{\varepsilon^2}{\delta^4}6(u_1)_{XXxx} - 2a_2u_1u_4 - 2a_2u_2u_3 \\ & -3a_3(u_1^2u_3 + u_1u_2^2) - 4\frac{\varepsilon}{\delta^2}(u_3)_{Xxxx} - 2\frac{\varepsilon}{\delta^2}P(u_3)_{Xx} - 4a_4u_1^3u_2 - a_5u_1^5\end{aligned} \tag{3.9}$$

With the ansatz

$$u_1 = A_1(X)e^{i\omega x} + \text{c.c.},$$

the equation (3.5) becomes

$$(\omega^4 - \omega^2 P + 1)A_1(X) \equiv \mu(\omega)A_1(X) = 0 \tag{3.10}$$

Using (2.15),

$$\mu(\omega)A_1(X) = \varepsilon^2 A_1(X) = \mathcal{O}(\delta^4), \tag{3.11}$$

which gives a contribution at higher order ($\mathcal{O}(\delta^5)$). Thus the $\mathcal{O}(\delta)$ equation is simply the identity $0 = 0$.

Solving the $\mathcal{O}(\delta^2)$ equation, we have

$$u_2 = A_0(X) + A_2(X)e^{2i\omega x} + \text{c.c.},$$

where

$$A_2(X) = -a_2A_1^2(X)/\mu(2\omega) \quad \text{and} \quad A_0(X) = -2a_2|A_1(X)|^2.$$

Thus we have at this order an expansion for u of the form

$$u = \delta A_1(X)e^{i\omega x} + \text{c.c.} + \delta^2 \left[A_0(X) + A_2(X)e^{2i\omega x} + \text{c.c.}\right].$$

Now considering the $\mathcal{O}(\delta^3)$ terms we have

$$\begin{aligned} \mathcal{L}u_3 &= \left[-\frac{\varepsilon}{\delta^2}(-4i\omega^3 A_1'(X) + 2iP\omega A_1'(X)) - 2a_2(A_2A_1^* + A_0A_1) - 3a_3|A_1|^2A_1\right] e^{i\omega x} \\ &\quad +(-2a_2A_1A_2 - a_3A_1^3)e^{3i\omega x} + \text{c.c.} \end{aligned} \tag{3.12}$$

To prevent resonance and consequent growth in u_3 we must avoid secular terms by ensuring that the right hand side of (3.12) is orthogonal to the homogeneous solution $e^{\pm i\omega x}$ of the linear problem $\mathcal{L}u = 0$. Although $\mathcal{L}e^{i\omega x} = \mu(\omega)e^{i\omega x} \neq 0$, it yields a right hand side which is $\mathcal{O}(\delta^4)$ from (3.11), and therefore at this order we may treat it as being the homogeneous solution. The secular terms can then only be zero to this order if

$$-\frac{\varepsilon}{\delta^2}(-4i\omega^3 A_1'(X) + 2iP\omega A_1'(X)) - 2a_2(A_2A_1^* + A_0A_1) - 3a_3|A_1|^2A_1 = 0. \tag{3.13}$$

Now, from their definitions and from (2.16)

$$-2a_2(A_2A_1^* + A_0A_1) - 3a_3|A_1|^2A_1 = 2\left[a_2^2\left(\mu(2\omega)^{-1} + 2\right) - 3a_3/2\right]|A_1|^2A_1 = 2\nu|A_1|^2A_1.$$

This expression gives our motivation for the choice of ν that we have taken in this paper. For the scaling range considered ν is $\mathcal{O}(\delta^2)$ and hence this expression does not play a role at $\mathcal{O}(\delta^3)$.

Thus, balancing terms of the same order, the solvability condition reduces to

$$2i\omega A_1'(X)[-2\omega^2 + P] = 0 \tag{3.14}$$

Therefore either $A'(X) = 0$, which corresponds to a periodic solution of (2.1), or if the solution is slowly varying (a homoclinic or heteroclinic solution) we must have

$$\omega^2 = \frac{P}{2} \tag{3.15}$$

to this order of asymptotics. This leading order relationship between ω^2 and P is central to our analysis, as it ties the derivation of the amplitude equation together with the condition that the Lagrangian energy is zero (see Section 5). Note that this expression, combined with (2.15) and the assumption that ε is small, implies that P is in the vicinity of 2 in the asymptotic analysis. Numerical computations suggest that it is accurate even when P is not asymptotically close to 2. In other words it is true not only for an asymptotically small neighbourhood.

If the scaling in (3.4) does not hold, that is, if

$$\nu = a_2^2\left(\mu(2\omega)^{-1} + 2\right) - 3a_3/2 = \mathcal{O}(1)$$

with respect to ε, the results then obtained correspond to the region of parameter space in which the orbits of interest are homoclinic and not heteroclinic connections. In this case, the expansion (3.3) is equivalent to taking $u \sim \varepsilon u_1 + \ldots$ with $x = \varepsilon X$, so that the quintic terms can be neglected in the amplitude equation, as discussed in the previous section (2.8)-(2.9).

Proceeding with the expansion, if we have (3.15) then the solvability condition for u_3 is satisfied for functions $A_1(X)$ for which $A_1'(X)$ is non-zero. Solving the resulting equation for u_3 then gives

$$u_3 = A_3(X)e^{3i\omega x} + \text{c.c.} \quad \text{with} \quad \mu(3\omega)\, A_3 = A_1^3(X)\left(\frac{2a_2^2}{\mu(2\omega)} - a_3\right).$$

Then the solvability condition is automatically satisfied for the $\mathcal{O}(\delta^4)$ equation for u_4, and we have simply that

$$u_4 = B_0 + B_2 e^{2i\omega x} + A_4 e^{4i\omega x} + \text{c.c.}$$

where B_0, B_2 and A_4 are all functions of A_1. Expressions for the coefficients B_0 and B_2, which are simply corrections to the terms A_0 and A_2, are given in the Appendix. We do not give an expression for the term A_4 here since it plays no further role in the derivation of the amplitude equation (3.16) for A_1.

At $\mathcal{O}(\delta^5)$ we obtain a solvability condition for u_5, avoiding secular terms, very much as in the calculation at $\mathcal{O}(\delta^3)$. This leads immediately to an envelope equation for A_1. After a bit of algebra, this takes the form

$$(6\omega^2 - P)\frac{\varepsilon^2}{\delta^4}A_1'' \quad = \quad \frac{\varepsilon^2}{\delta^4}A_1 - 2\frac{\nu}{\delta^2}A_1|A_1|^2 + i\frac{\varepsilon}{\delta^2}d|A_1|^2A_1'(X) + c|A_1|^4A_1$$

Substituting the earlier solvability condition $\omega^2 = P/2$ into this equation we have,

$$-(6\omega^2 - P)\frac{\varepsilon^2}{\delta^4}A_1'' + \frac{\varepsilon^2}{\delta^4}A_1 - 2\frac{\nu}{\delta^2}|A_1|^2A_1 + c|A_1|^4A_1 + i\frac{\varepsilon}{\delta^2}dA_1'|A_1|^2 = 0. \tag{3.16}$$

The coefficients c and d are also given in the Appendix. If there is no quadratic term in $f(u)$ ($a_2 = 0$), then $d = 0$.

The derivation of the amplitude equation, which includes solvability conditions at several orders of the perturbation expansion, is analogous to the resolution of a singular Hopf bifurcation [1]. Here the choice of the small parameters δ, ε and ν leads to both a local description, valid very close to $P = 2$, and a nonlocal description, valid for larger values of $2 - P$ corresponding to the balance of linear and nonlinear terms in (3.16).

3.1 Analysis of the amplitude equation

We now analyse the equation (3.16). If we look for solutions of the form

$$A_1(X) = r(X)e^{i\varphi(X)},$$

with $r(X)$ real, then

$$\varphi_X(X) \quad = \quad \frac{\delta^2}{\varepsilon}\left(\frac{dr^4}{4} + c_1\right)\frac{1}{(6\omega^2 - P)r^2} \tag{3.17}$$

$$(6\omega^2 - P)\frac{\varepsilon^2}{\delta^4}r_{XX} \quad = \quad \frac{\varepsilon^2}{\delta^4}r - 2\frac{\nu}{\delta^2}r^3 + gr^5, \tag{3.18}$$

where

$$g = c - \frac{3d^2}{16(6\omega^2 - P)}\frac{\varepsilon^2}{\delta^4}. \tag{3.19}$$

If we express the equation (3.18) in unscaled variables by setting $R = \delta r$ and $x = X/\varepsilon$ then we obtain (2.8).

In general, (3.18) has a fixed point at $r = 0$ and a homoclinic solution connecting $r = 0$ to itself. This orbit satisfies the identity obtained by integrating r_X times (3.18) which is given by

$$\frac{6\omega^2 - P}{2}\frac{\varepsilon^2}{\delta^4}r_X^2 = \frac{\varepsilon^2}{2\delta^4}r^2 - \frac{\nu}{2\delta^2}r^4 + \frac{g}{6}r^6. \tag{3.20}$$

Setting $r_X = 0$ in this expression we have that the maximum amplitude r_H of the homoclinic solution is given by

$$r_H^2 = \frac{3\nu}{2g\delta^2}\left[1 - \sqrt{1 - \frac{4g\varepsilon^2}{3\nu^2}}\right] \tag{3.21}$$

provided that $\varepsilon^2 < 3\nu^2/4g$.

The amplitude equation also has two non-zero fixed points ($r_{XX} = 0$) at points $r_0^\pm > 0$ satisfying the quartic equation

$$\left(r_0^\pm\right)^2 = \frac{\nu}{g\delta^2}\left[1 \pm \sqrt{1 - \frac{g\varepsilon^2}{\nu^2}}\right]. \tag{3.22}$$

Observe that these fixed points correspond to those periodic solutions of the underlying system with frequency $\omega^2 = P/2$ to leading order. It is not difficult to show that

$$r_0^- < r_H \le r_0^+$$

with equality on the right hand side occurring precisely when

$$\varepsilon^2 = 3\nu^2/4g. \tag{3.23}$$

At this point there is a heteroclinic connection from $r = 0$ to the fixed point r_0^+ and for which

$$r_0^2 = r_H^2 = \frac{3\nu}{2g\delta^2}. \tag{3.24}$$

Using (2.15), (2.16) the condition (3.23) can be written in terms of P and the original coefficients a_j of $f(u)$ (2.1),

$$\mu(\omega)g \;=\; \frac{3}{4}\nu^2 = \frac{3}{4}\left[\frac{3}{2}a_3 - a_2^2\left(\mu(2\omega)^{-1} + 2\right)\right]^2. \tag{3.25}$$

This is the condition for the existence of a heteroclinic connection of frequency $\omega^2 = P/2$ and thus it gives the Maxwell load $P = P_M$, and corresponding heteroclinic connection $u_M(x)$. In Section 8 we give a simplified formula for P_M when P is very close to 2.

The equation (3.17) provides a phase correction to the periodic solution, which is a higher order correction to the condition $\omega^2 = P/2$ (3.15) consistent with the Maxwell load condition $L = 0$. From equation (3.17) we see that on a homoclinic orbit with $r \to 0$ the equation for φ becomes singular unless we set $c_1 = 0$. Doing this and integrating the resulting equation in a neighbourhood of the fixed point on the heteroclinic orbit we have

$$\varphi = \frac{\delta^2}{\varepsilon} \frac{dr_0^2}{4(6\omega^2 - P)} X + \varphi_0 \tag{3.26}$$

where φ_0 is a constant. Again observe that $\varphi = \varphi_0$ if $a_2 = d = 0$. Substituting $\omega^2 = P/2$ and (3.24) we have

$$\frac{\delta^2}{\varepsilon} \frac{dr_0^2}{4(6\omega^2 - P)} = \frac{-9a_2^2 \nu}{g\mu(2\omega)^2} \tag{3.27}$$

Thus, including the phase correction and $X = \varepsilon x$ in the periodic solution $u(x)$ we get the corrected frequency ω,

$$\begin{aligned} u(x, X) &\sim \delta A_1(X) e^{i\omega x} + \text{c.c.} = \delta r_0 e^{i\varphi(X) + i\sqrt{P/2}x} + \text{c.c.} = \delta r_0 e^{i(\sqrt{P/2} + \nu\omega_1)x} + \text{c.c.} \\ \omega &= \sqrt{\frac{P}{2}} + \nu\omega_1 = \sqrt{\frac{P}{2}} - \frac{9a_2^2\nu}{\mu(\sqrt{2P})^2 g}. \end{aligned} \tag{3.28}$$

Note that the phase correction ω_1 is defined in terms of ω, so that an approximation to the phase correction is obtained by setting $\omega = \sqrt{P/2}$ in the definition of ω_1. We will show in Section 5 that (3.28) is equivalent to setting $L = 0$ to this order of asymptotics.

In Section 7 we compare these results to the numerics, applying the general formulas of this section to the particular case $a_2 = -1$, $a_4 = a_5 = 0$.

4 The case of the symmetric nonlinearity

As we have seen above, the condition (3.25) is a complicated function of the coefficients a_j and P. However, this expression simplifies greatly in the symmetric case when $a_2 = a_4 = 0$, $a_3 < 0$ and we consider (after a preliminary rescaling (2.12) by $a_5^{-1/4}$) the solutions of

$$\mathcal{L}u + \frac{a_3}{\sqrt{a_5}} u^3 + u^5 = 0. \tag{4.1}$$

Furthermore, the corrections to the asymptotic expansion are at higher order and thus the asymptotic expression is more accurate, as demonstrated by the numerical calculations.

In this case, we use the scaling (2.12) to get (2.13) and follow the procedure outlined in the previous section, with $\delta = \varepsilon/\nu^{1/2}$ and ε and ν given by (2.15) and (2.18). Then the zero mode A_0 and the higher mode $A_2 e^{2i\omega x}$ which are proportional to a_2 and a_4 vanish, and the symmetric solution has a much simpler modal expansion of the form

$$u(x, X) = \delta A_1(X) e^{i\omega x} + \delta^3 A_3(X) e^{3i\omega x} + \ldots$$

In this calculation ν is chosen to give an equation exactly of the form (3.18) with

$$d = 0 \quad \text{and} \quad g = 10 - 3a_3^2/(a_5\mu(3\omega)).$$

Observe that as $d = 0$ there is no $\mathcal{O}(\varepsilon)$ 'phase correction' to be made for ω. The analysis of the amplitude equation proceeds exactly as before, with an evolution of a homoclinic connection into a heteroclinic one. The Maxwell condition (3.25) then reduces to

$$\mu(\omega)a_5 \quad = \quad a_3^2\left[\frac{27}{160}+\frac{3\mu(\omega)}{10\mu(3\omega)}\right]. \tag{4.2}$$

where μ is given by (2.15) and $\omega^2 = P/2$. An analogous result for wave number selection in the Swift-Hohenberg equation with symmetric nonlinearity [3], [17] is valid for P near 2, in the notation of this paper.

It is interesting to note that the critical curve (4.2) is just the limit of (3.25) as $a_2, a_4 \to 0$ for the general non-symmetric case ($a_2 \neq 0, a_5 \neq 0$), and it is not necessary to do a different analysis. Then the equation (3.25) for the Maxwell load $P = P_M$ has the form

$$\left[10a_5 - \frac{3a_3^2}{\mu(3\omega)} + \mathcal{O}(a_2, a_4)\right]\mu(\omega) \quad = \quad \frac{3}{4}\left[\frac{3}{2}a_3 - a_2^2(\mu(2\omega)^{-1}+2)\right]^2 \tag{4.3}$$

where

$$\mu(\omega) = 1 - \frac{P_M^2}{4} \quad \text{for} \quad \omega^2 = \frac{P_M}{2} \tag{4.4}$$

which gives (4.2) as $a_2 \to 0$.

Furthermore, from (4.3) we see that if $a_2 \neq 0$, then the critical load P_M approaches 2 when $a_3 \to a_2^2 h(P_M)$, whereas if $a_2 = 0$ then P_M approaches 2 when $a_3^2/a_5 \to 0$.

5 The Lagrangian condition $L = 0$

The differential equation (2.1) is the first variation of the Lagrangian energy

$$L = \int\left[\frac{u''^2}{2} - \frac{Pu'^2}{2} + \frac{u^2}{2} + F(u)\right]\,dx, \quad F'(u) = f(u).$$

Whilst this integral is not defined in general for a periodic solution, we may consider the energy per period T as

$$L = \int_0^T\left[\frac{u''^2}{2} - \frac{Pu'^2}{2} + \frac{u^2}{2} + F(u)\right]\,dx.$$

In [19] it is shown that periodic solutions satisfying $L = 0$ play a significant role in the behaviour of the underlying system. In particular, adjoining such a solution to a localised solution does not change its energy, with obvious implications to any minimisation argument. Furthermore, at the Maxwell load, the periodic solution is the limit of a heteroclinic connection. This has a constant Hamiltonian of $H=0$ and hence the periodic solution must also satisfy $H=0$ so that

$$u'''u' - \frac{u''^2}{2} + P\frac{u'^2}{2} + \frac{u^2}{2} + F(u) = 0.$$

Integrating this expression over one period, and applying integration by parts, we deduce that if u is a periodic solution with zero Hamiltonian then

$$\int_0^T \left[-\frac{3}{2}u''^2 + P\frac{u'^2}{2} + \frac{u^2}{2} + F(u) \right] \, dx = 0.$$

Hence, the condition $L=0$ is equivalent to

$$\int_0^T \left[2u''^2 - Pu'^2 \right] \, dx = 0. \tag{5.1}$$

Now suppose that u has a modal expansion of the form

$$u = \delta^2 A_0 + \delta A_1 e^{i\omega x} + \delta^2 A_2 e^{2i\omega x} + \delta^3 A_3 e^{3i\omega x} + \text{c.c.}.$$

Substituting into the identity (5.1) we have

$$2\omega^4 \left[\delta^2 |A_1|^2 + 16\delta^4 |A_2|^2 + 81\delta^6 |A_3|^2 .. \right] - P\omega^2 \left[\delta^2 |A_1|^2 + 4\delta^4 |A_2|^2 + 9\delta^6 |A_3|^2 .. \right] = 0.$$

Hence, ω must satisfy the condition

$$\omega^2 = \frac{P}{2} \left(\frac{|A_1|^2 + 4\delta^2 |A_2|^2 + 9\delta^4 |A_3|^2 ..}{|A_1|^2 + 16\delta^2 |A_2|^2 + 81\delta^4 |A_3|^2 ..} \right) = \frac{P}{2} \left(1 - 12\delta^2 \left| \frac{A_2}{A_1} \right|^2 \right) + \mathcal{O}\left(\delta^4\right).$$

Substituting for A_1 and A_2 the values taken at the Maxwell load gives

$$\omega^2 = \frac{P}{2} \left(1 - 18\nu \frac{a_2^2}{g\mu (2\omega)^2} \right) + \mathcal{O}\left(\delta^4\right)),$$

which is the same to order $\mathcal{O}(\delta^4)$ as the phase correction to ω (3.28) derived from the amplitude equation. Observe that in the symmetric case we have simply

$$\omega^2 = \frac{P}{2} + \mathcal{O}\left(\delta^4\right).$$

The conditions $L=0$ and $H=0$ were used in [17] to describe wave number selection in a Swift-Hohenberg model corresponding to the symmetric case of this paper (2.6).

6 The condition $H=0$.

The equation

$$u'''' + Pu'' + u + f(u) = 0$$

has a first integral

$$H = u'u''' - \frac{1}{2}u''^2 + \frac{P}{2}u'^2 + F(u), \quad F'(u) = f(u). \tag{6.1}$$

As observed above, localised solutions of the system (homoclinic connections) must satisfy the condition $H = 0$. There may also be periodic solutions which do not satisfy this condition. We now show that for the problems considered in this paper the condition that a periodic solution satisifies $H = 0$ is equivalent (to the level of asymptotics considered) to the condition that there should be a heteroclinic connection between zero and the fixed point r_0 in the amplitude equation. If we consider the underlying differential equation and substitute a purely periodic solution of the form

$$u = \delta^2 A_0 + \delta A_1 e^{i\omega x} + \delta^2 e^{2i\omega x} + c.c.$$

then, comparing terms in $e^{ik\omega x}$ and substituting where appropriate, then the condition satisfied when $k = 1$ leads to the non differentiated terms in the amplitude equation. In particular the (linear) coefficient of A_1 is given by $\mu(\omega)A_1$ with

$$\mu(\omega) = \omega^4 - P\omega^2 + 1.$$

In contrast we may substitute the pure periodic solution into the integrated equation (6.1). After some manipulation we find that the resulting expression has a contribution of the form $2\lambda(\omega)|A_1|^2/2 + h.o.t.$ where

$$\lambda(\omega) = -3\omega^4 + P\omega^2 + 1.$$

If we compare λ and μ we see that they are equal iff $\omega^2 = P/2$, precisely as given by the asymptotic calculation.

Now consider the nonlinear terms. For simplicity we consider the case of a symmetric nonlinearity only so that $u(x)$ has a single mode expansion of the form $u(x) = A_1 e^{i\omega x}$+c.c.. A simple calculation shows that if n is odd then the coefficient of $e^{i\omega x}$ in $u(x)^n$ is given by $C^n_{(n-1)/2} A_1^n$. Similarly, the constant term in the expression $u(x)^{n+1}/(n+1)$ is given by $2C^n_{(n-1)/2}|A_1|^{n+1}/(n+1)$.

We conclude that if the underlying differential equation leads to an equation for A_1 of the form

$$\sum \alpha_k A_1^k = 0,$$

then, provided $\omega^2 = P/2$ the equation for H is simply

$$H = 2\sum \alpha_k |A_1|^{k+1}/(k+1).$$

Thus, if (as before) $A_1 = re^{i\varphi}$ then the equation for H is twice the first integral of the resulting amplitude equation for r. Hence the condition that this integral is zero is equivalent to the condition that $H = 0$ in this case.

7 Numerical calculations

The main conclusions of the previous analysis is that the amplitude equation predicts the existence of homoclinic solutions of frequency close to $\omega^2 = P/2$ which converge towards a heteroclinic connection to a periodic solution at the Maxwell load P_M. In this section we present some numerical results for the two cases of the non-symmetric nonlinearity

($a_2 \neq 0$) and the symmetric nonlinearity ($a_2 = a_4 = 0$). Our results look at two features of the solution described above. Firstly we look at the calculation of the periodic solution associated with the Maxwell load, where we make a comparison between the asymptotic formula (3.25) for the critical Maxwell load P_M in terms of the parameters a_j of the nonlinear equation and the computations both of the periodic solution and the associated loads. An interesting conclusion from our numerical computations is that the asymptotic formulae appear to remain accurate even when the values of δ, ε and ν are not particularly small. Secondly, we will calculate the true homoclinic solutions of (2.1) as functions of the load P and then compare these with the solutions as predicted by the asymptotic analysis.

7.1 Periodic solutions

7.1.1 Calculation of the solution branches

We look for symmetric solutions in x of the equation (2.1) which satisfy the periodic boundary conditions

$$u(0) = u(T), \quad u'(0) = u'(T)$$

where $T = 2\pi/\omega$. There are many such periodic solutions and to find the ones of interest we specify certain additional conditions. Firstly we fix the *phase* of the solution so that

$$u'(0) = u'(T) = 0.$$

Secondly we look only at solutions symmetric about $x = 0$ so that $u(x) = u(-x)$ and hence

$$u'''(0) = u'''(T) = 0.$$

It is certainly true that as the equation itself admits the symmetry $u \to u, \quad x \to -x$ there is a class of symmetric periodic solutions which bifurcate from zero at the critical value of $P = 2$ and which retain this symmetry. Thus it is not being overly restrictive to confine our attention to this class. To fix things we need one additional condition. Two are of interest, the condition for a zero Lagrangian $L = 0$ solution (2.3) which we call condition L, and secondly the condition for a zero *Hamiltonian* $H = 0$ defined by (2.4) which we call condition H. These are chosen because a solution satisfying condition L will correspond to our earlier analysis and condition H is necessary for a periodic solution to have a heteroclinic connection to the origin.

In principle, given a suitably chosen P, then either of the two conditions L or H lead to a well defined periodic solution $u(x)$ together with an associated period T and frequency ω. A path of such solutions may then be computed numerically. A simple procedure which works well for solutions satisfying condition L is firstly to guess values of $u(0)$, T and H. Using (2.4) we may then determine $u''(0)$ and we have $u'(0) = u'''(0) = 0$. We then integrate the system (2.1) together with the equation

$$\frac{dL}{dx} = \frac{u''^2}{2} - P\frac{u'^2}{2} + \frac{u^2}{2} + F(u)$$

as an initial value problem over the interval $[0, T]$ using a variable step Runge-Kutta Mersen method with error tolerance of 10^{-12}. Using the Powell-hybrid nonlinear solver

SNSQE (see `http://gams.nist.gov/serve.cgi/Module/NMS/SNSQE/2828`), the values of $u(0)$, T and L are then adjusted until the three conditions

$$u(0) - u(T) = 0, \quad u'(T) = 0 \quad \text{and} \quad L(T) = 0 \tag{7.1}$$

are satisfied. The procedure for finding solutions satisfying condition H is very similar but slightly simpler as in this case only $u(0)$ and T are to be determined. These procedures worked well given reasonable initial guesses. Having determined the solution at one value of P a branch of such solutions is then computed by using Keller's pseudo arc-length method [13].

When the solution satisfies the condition H elementary bifurcation theory shows that there is a path of periodic solutions which bifurcates for $P < 2$ from the zero solution at the critical load value of $P = 2$. This curve has a fold bifurcation at a point $P_1 < 2$ and exists for $P > P_1$. In contrast the curve satisfying condition L does not bifurcate from the zero solution; however it does have a fold bifurcation precisely at the Maxwell load $P = P_M$ where it *also* satisfies the condition H [19]. We present below two numerically computed curves of such periodic solutions for the case of the non-symmetric nonlinearity

$$f(u) = -u^2 + u^3/2.$$

In Figures 7.1 and 7.2 we take solutions satisfying condition L and plot the frequency ω and Hamiltonian H of the solution in the first figure and the amplitude $u(0)$ in the second. Observe that the limit points in Figure 7.1 occur precisely when $H = 0$ and that at the left most limit point we also have $\omega^2 \approx P/2$.

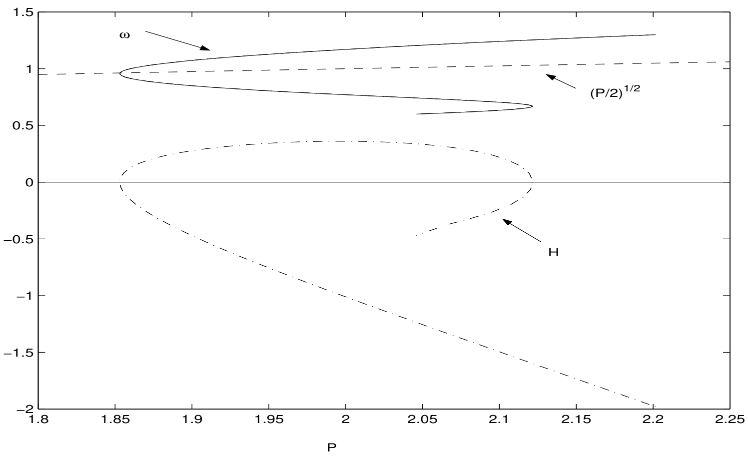

Figure 7.1: *H and ω on the branch of the periodic solution satisfying $L = 0$*

In Figures 7.3,7.4 and 7.5 we plot those periodic solutions satisfying condition H giving ω and L and $u(0)$ in this case. We use these results below to identify the Maxwell load P_M.

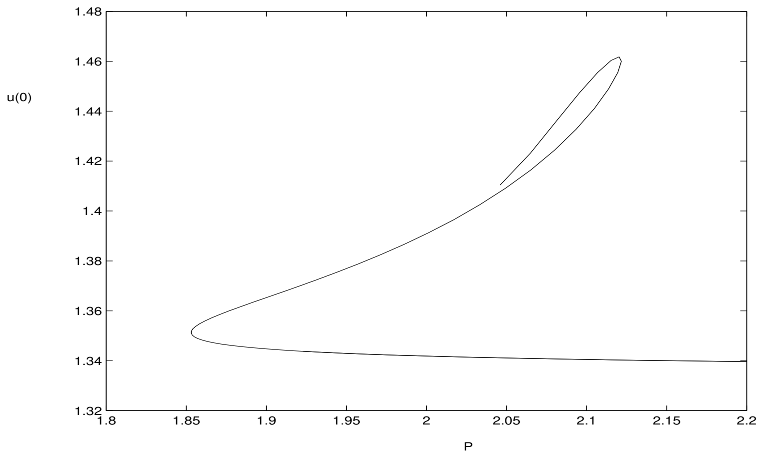

Figure 7.2: $u(0)$ *on the branch of the periodic solution satisfying* $L = 0$

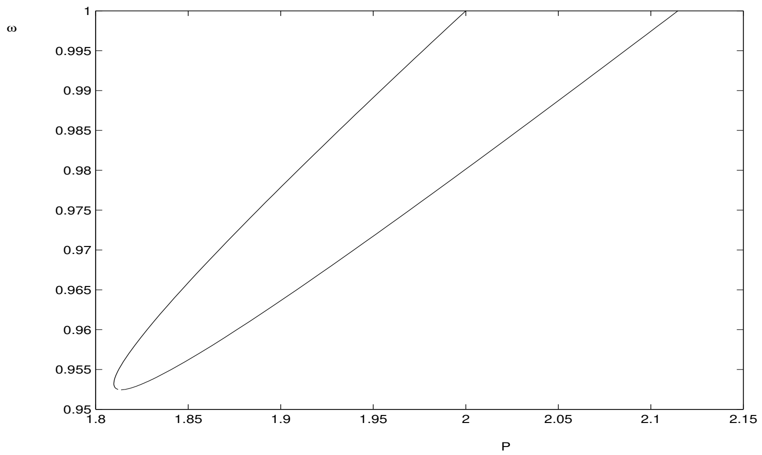

Figure 7.3: ω *on the branch of periodic solutions satisfying* $H = 0$

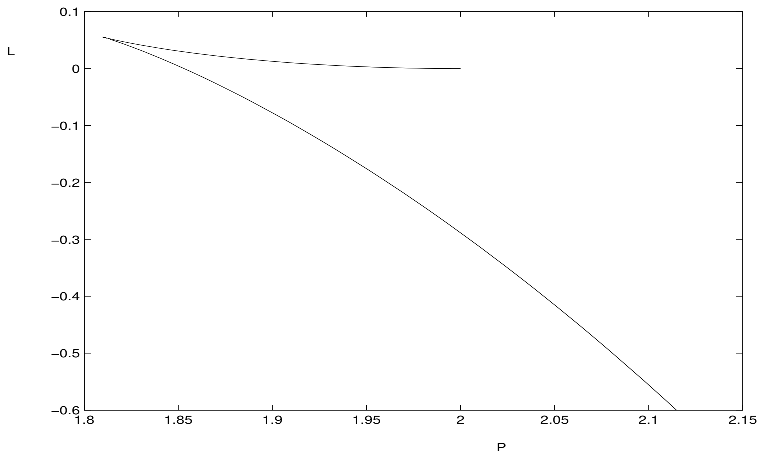

Figure 7.4: L *on the branch of periodic solutions satisfying* $H = 0$

7.1.2 Calculation of the Maxwell load

In the above figures it is clear that there are well defined values of P where both condition L and condition H are satisfied simultaneously. This is precisely the Maxwell condition

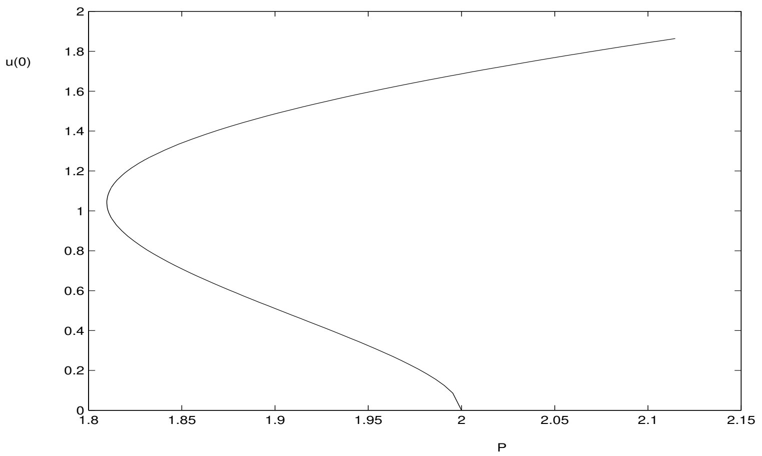

Figure 7.5: $u(0)$ *on the branch of periodic solutions satisfying* $H = 0$

for the load $P \equiv P_M$ identified earlier, at which the amplitude equation has a heteroclinic orbit with zero Hamiltonian connecting the zero solution to the periodic solution given by the fixed point. Such points are locally unique, so that P_M and the associated functions u_M and ω_M are locally unique functions of the coefficients of the function $f(u)$. We may thus identify curves of such points at the Maxwell load parametrised by these coefficients.

To compute such curves, we modify the method described earlier so that $H = 0$ is given and $P, u(0)$ and T are unknowns. The initial value problem is then integrated as before and $P, u(0)$ and T are determined so that the condition (7.1) is satisfied. Branches of solutions are then found as before.

To present our results we consider first the asymmetric case of

$$f(u) = -u^2 + a_3 u^3 \tag{7.2}$$

In Figures 7.6,7.7 and 7.8 we plot P_M, ω_M and $u_M(0)$ as functions of a_3. The computed solutions presented as solid lines are compared with the asymptotic values obtained in Section 3 which are presented as dashed lines. The periodic solution $u_M(x)$ is given by

$$u_M(x) = \delta^2 A_0 + \delta^4 B_0 + \delta A_1 e^{i\omega x} + (\delta^2 A_2 + \delta^4 B_2) e^{2i\omega x} + \delta^3 A_3 e^{3i\omega x} + .. + \text{c.c.}$$

with $\delta A_1 = \sqrt{\frac{3\nu}{2g}}$ (3.24) and A_j and B_j given in Section 3 and the Appendix.

The agreement here is reasonable for all values of a_3 (the value of ω_M is particularly well approximated), and the agreement is especially good for a_3 close to 38/27 where we may use the simplified expansions demonstrated in Section 8.

The small discrepencies arise from the sensitivity of $u(0)$ to higher order asymptotic contributions from the zero mode and the second harmonic. We anticipate some improvement could be realised through an additional rescaling based on ν.

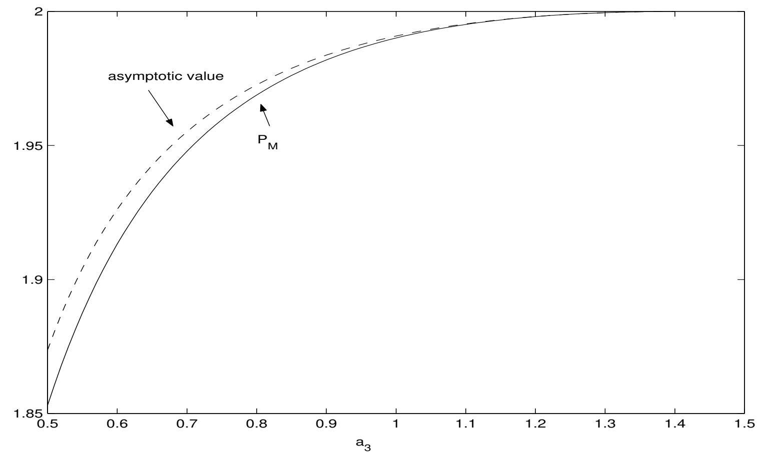

Figure 7.6: *Numerical (solid) and asymptotic (dashed) calculations of* P_M.

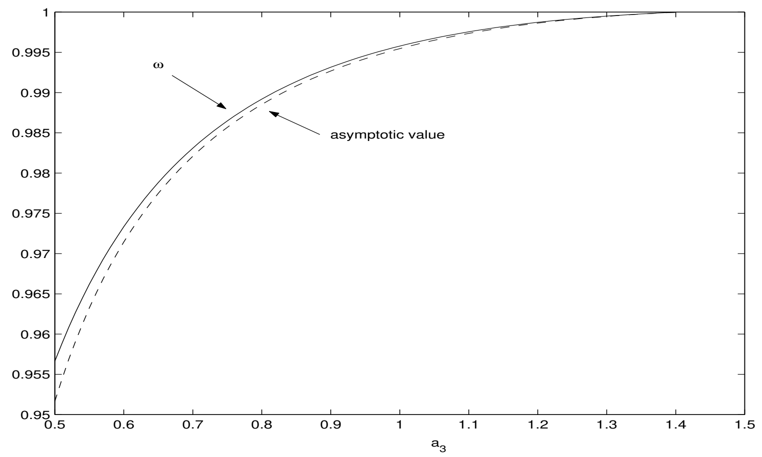

Figure 7.7: *Numerical (solid) and asymptotic (dashed) calculations of* ω_M

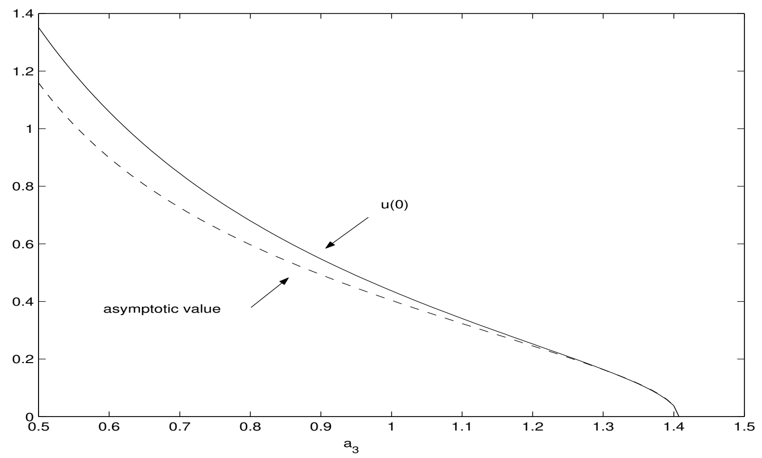

Figure 7.8: *Numerical (solid) and asymptotic (dashed) calculations of* $u_M(0)$

We now consider the symmetric problem,

$$f(u) = a_3 u^3 + u^5, \quad a_3 < 0,$$

discussed in Section 4. In Figures 7.9, 7.10 and 7.11 we again plot P_M, ω_M and $u_M(0)$ over a range of values of a_3 and include the asymptotics as determined by the formulae given in Section 4. Here the amplitude of the periodic solution is given by

$$a_5^{1/4}\delta A_1 = a_5^{1/4}\sqrt{\frac{3\nu}{2g}} = \frac{1}{2}\sqrt{\frac{-9a_3}{g}}, \tag{7.3}$$

$$g = 10 - \frac{3a_3^2}{a_5\mu(3\omega)} = 10 - O(\nu^2) \tag{7.4}$$

At the Maxwell load $P = P_M$, the heteroclinic solution of (4.1) takes the form

$$u(x) = \sqrt{\frac{-9a_3}{g}\frac{1}{1+e^{\theta\varepsilon(x-x_0)}}}\cos\left(\sqrt{\frac{P_M}{2}}x + \varphi_0\right) \tag{7.5}$$

where $\theta = \sqrt{2/P_M}$, x_0 is arbitrary and φ_0 is an undetermined phase, the calculation of which requires asymptotics beyond all orders.

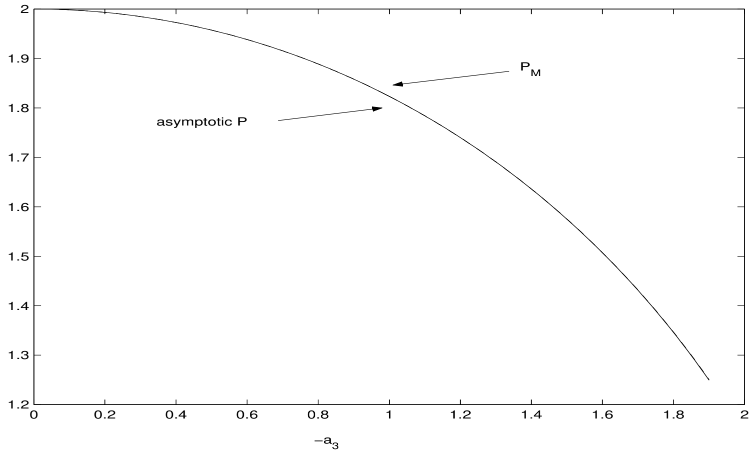

Figure 7.9: *Numerical (solid) and asymptotic (dashed) calculations of P_M in the symmetric case.*

The agreement between the computed values and the predictions of the asymptotic formulae (7.3)-(7.5) is truly remarkable given the wide range of values of a_3 considered, which include values which are not particularly small. Indeed the two curves of numerical and asymptotic results cannot be distinguished from each other. Note that this agreement is better than for the non-symmetric problem, because in this case $u(x)$ is approximated well by a single mode.

Above we have shown good agreement between the asymptotic approximation and numerical calculation of P_M. In Section 2 we outlined the scaling which is the basis for the derivation of the amplitude equation. For the asymmetric case shown in Figures

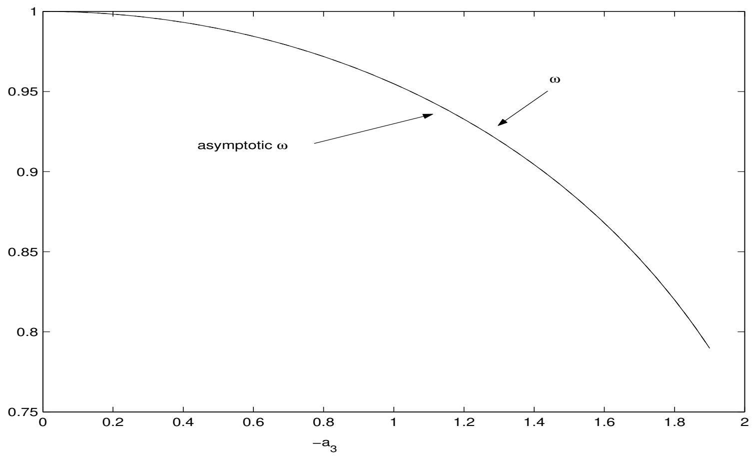

Figure 7.10: *Numerical (solid) and asymptotic (dashed) calculations of* ω_M *in the symmetric case.*

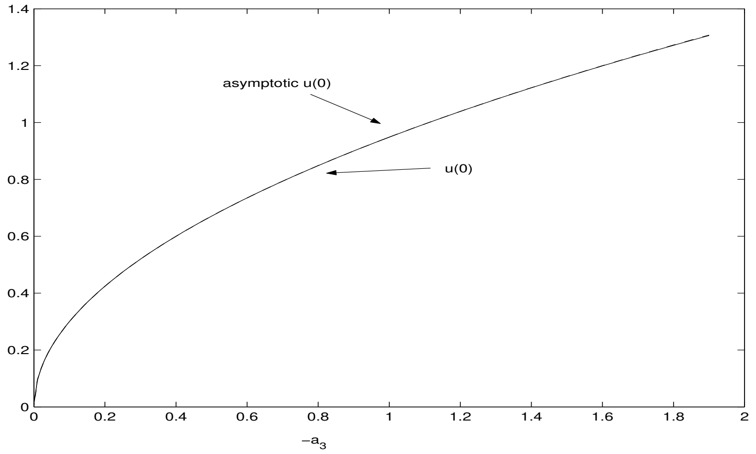

Figure 7.11: *Numerical (solid) and asymptotic (dashed) calculations of* $u_M(0)$ *in the symmetric case.*

7.6–7.8, the range of parameters corresponds to $0 < \varepsilon^2 < 0.2$ and $0 < \delta < 0.4$. For the symmetric case shown in Figure 7.9–7.11, the parameters correspond to $0 < \varepsilon^2 < 0.7$ and $0 < \delta < 0.55$. While this appears to be a fairly large range for these parameters, the success of the asymptotic approximation is due to several key points. The choice of the small parameters is motivated by the appropriate balance of linear and nonlinear terms, not only a linear analysis around the critical bifurcation point. The amplitude equation, whose derivation relies on a small amplitude scaling δ, is appropriate for a range of parameters for which this balance holds, thus providing a reasonable approximation to the Maxwell load for a larger range of parameters. The frequency ω, which is a crucial element in describing the behavior of localized solution, is determined through a solvability condition, rather than by scaling about the frequency $\omega = 1$ at the critical load $P = 2$. In addition, since the coefficient of δ^j is $(A_1)^j = O(1)$ in the approximation to the solution u, we expect that we can find asymptotically valid approximations in which u is actually close to unity. This is shown in Figures 7.8 and 7.11. As P decreases so that $2 - P$ is no longer small in the asymptotic sense defined above, the qualitative behavior of the solution moves towards kinks [12], and an analysis based on the amplitude equation is no longer valid.

7.2 Calculations of the homoclinic solution

The homoclinic solutions of the amplitude equation correspond to homoclinic solutions of the underlying fourth order equation. It is possible to find numerical solutions of these and then follow them as coefficients are varied using the techniques described in [12]. The equation (2.1) is solved by collocation on a large interval, and is matched to boundary conditions derived from the form of the homoclinic solution. For $u_H(x)$ such a homoclinic solution, we can characterise this by the value of $u_H(0)$ and by the *end shortening*

$$E = \frac{1}{2}\int_{-\infty}^{\infty} u'^2 dx.$$

In the following figures, and for all of the subsequent calculations in this Section, we plot E and $u_H(0)$ for the case of $a_3 = -\sqrt{2}$,

$$f(u) = -\sqrt{2}u^3 + u^5 \tag{7.6}$$

looking for solutions parametrised by P. Note that a_3^2/a_5 is not small in this case.

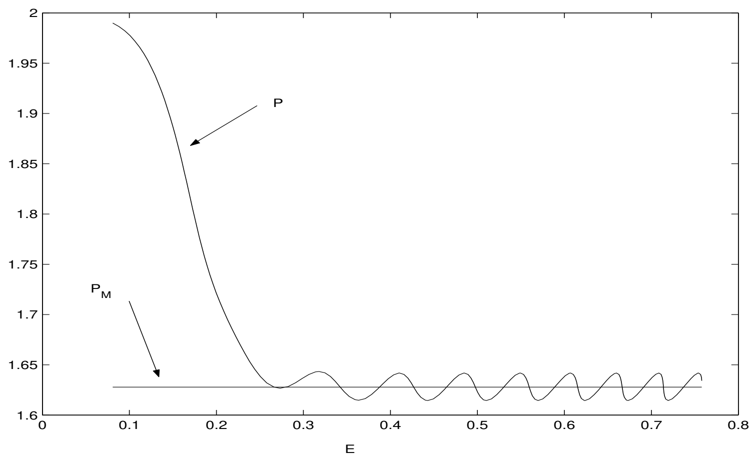

Figure 7.12: *P as a function of end-shortening, showing the Maxwell load P_M*

For the symmetric problem (7.6) we can compare the numerically computed value of $u_H(0)$ with the value of $r_H(0)$ determined in (3.21) given by

$$\delta^2 r_H^2(0) = \frac{-9a_3}{10}\left(1 - \sqrt{1 - \frac{160\varepsilon^2}{27a_3^2}}\right),$$

where $\varepsilon^2 = 1 - P^2/4$. The resulting graphs are given in Figure 7.13. Observe that the curve derived from the amplitude equation closely approximates the numerically computed curve, but that the latter curve has a small oscillatory variation about it close to the Maxwell load, so that in the computations P can take values slightly lower than P_M, and P oscillates between values P_1 and P_2 as $u(0) \to u_M(0)$.

The oscillations of P about P_M visible in Figure 7.13 arise from terms beyond all orders in the asymptotic expansion not considered in this paper and the nature and origin of these

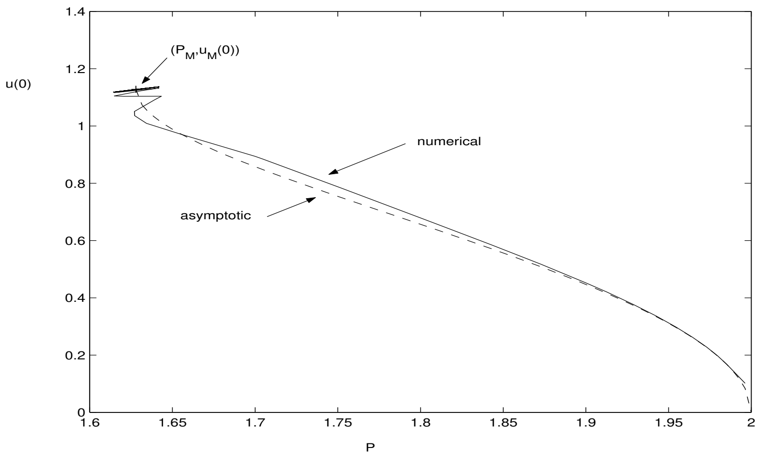

Figure 7.13: *The asymptotic amplitude (dashed) of the homoclinic solution compared with the numerical solution (solid), showing the oscillatory behaviour close to the Maxwell load.*

oscillations is described in [3],[17],[12]. The limiting value of $P = P_M$ at which the curve r_H ceases to exist is approximately located at the mid-point $(P_1+P_2)/2$ of the oscillations indicated in Figure 7.13. In Figure 7.14 we plot this mid-point as a function of a_3, together with the values P_1 and P_2 corresponding to the maximum and minimum values of P in the oscillatory part of the graph close to the Maxwell load. We can then compare these values with the computed value of P_M. The agreement between the computed value of the midpoint and the Maxwell load P_M is excellent for a wide range of values of a_3 none of which is especially small. For the symmetric case, asymptotic calculations reported in [3],[17] imply that as $\nu \to 0$ the width $P_1 - P_2$ of the oscillations about P_M is of $O(e^{-\mathcal{C}/\nu^2})$ for a constant $\mathcal{C} > 0$. A more refined calculation, which we detail in a future paper [4], indicates that as $a_3 \to 0$

$$P_1 - P_2 \sim \frac{1}{2\pi} a_3^4 e^{-2\pi/a_3^2}. \tag{7.7}$$

In Figure 7.14 we plot both the numerical values (solid) of P_1, P_2 and the mid-point $(P_1 + P_2)/2$ and compare these with the theoretical values of P_M given earlier and of P_1 and P_2 derived from (7.7). Again the agreement between theory and numerics is excellent. This indicates that although P can be less than P_M, the minimum value of P is always exponentially close (in the sense of (7.7) to P_M.

At the point $P = P_M$ the amplitude equation has a heteroclinic rather than a homoclinic solution. This is a very good approximation of each half of the limiting homoclinic solution in the limit of $E \to \infty$. Now at the same value of P_M the underlying equation has an infinite number of homoclinic solutions of increasing end-shorterning which tend towards a heteroclinic solution [18]. For comparison in Figure 7.15 we take a homoclinic solution which has 11 significant oscillations of the periodic solution in the centre portion. The numerical calculation gives one half of this solution, so that the origin is placed at a mid-point for which $u'(0) = 0$. This we compare with the modulated heteroclinic solution $u_H(x)$ calculated using (7.5) with $\varepsilon = \sqrt{1 - P_M^2/4}$ and $\theta = \sqrt{2/P_M}$. In this solution the phase φ_0 of the oscillatory term is fixed to be 0 by the condition that $u'(0) = 0$. However, we have freedom over the choice of the point x_0. A very good fit is obtained by taking $x_0 = 35.3$. Observe that the amplitude, frequency and phase of the asymptotic solution closely match that of the numerically computed solution. The asymptotic estimation of

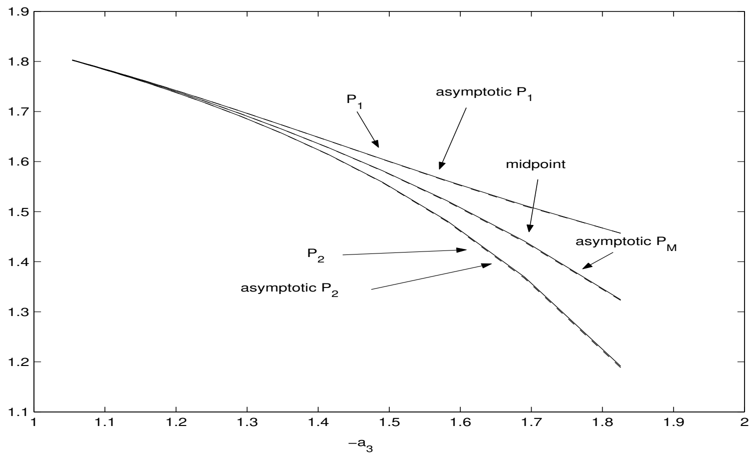

Figure 7.14: *A comparison of the theoretical Maxwell load P_M and P_1 and P_2 (dashed) with the mid-point and P_1 and P_2 (solid) of the numerically computed oscillations.*

x_0 requires the use of exponential asymptotics and is left for a future paper.

Comparisons with numerics reveal the extent of the influence of "beyond-all-order" terms which make up the difference between the true result and the asymptotic portrayal. In particular, the beyond all orders terms help to determine the phase of the solution and predict the existence of solutions in an exponentially close neighbourhood of P_M as $a_3 \to a_2^2 h(P)$ or as $a_3/\sqrt{a_5} \to 0$. A true comparison with the homoclinic requires looking beyond the integrable case to asymptotics beyond all orders and we leave this to a future calculation.

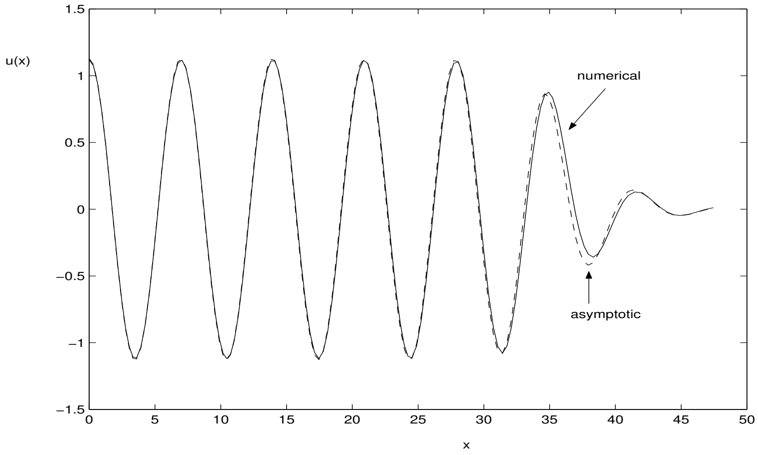

Figure 7.15: *The heteroclinic solution computed (solid) compared with the asymptotic (dashed) calculation.*

8 Comparison with previous analyses

We focus on several previous analyses which were primarily based on taking P very close indeed to 2. In [22] an amplitude equation was derived and in [12] a normal form analysis was used for the case of $a_2 = -1$, $a_3 > 0$ and $a_4 = a_5 = 0$. In both cases the significant value of $a_3 = 38/27$ was identified as the global stabilisation limit as $P \to 2$ leading to the heteroclinic connection.

Our analysis also gives this special value, under the assumption that $2 - P$ is a quadratic function of ν,

$$P \sim 2 - p\nu^2 \text{ for } \nu \ll 1. \tag{8.1}$$

Such a quadratic relationship between $2 - P$ and ν follows from an asymptotic approach for the amplitude equation based solely on the proximity of P to the bifurcation point, as discussed in Section 2.2.

Then, by the definition of ε and μ (2.14)-(2.15) and ω from (3.15) and (3.28) we have

$$\begin{aligned} \mu(2\omega) &\sim 9 + O(\nu) \\ \omega^2 &= \frac{P}{2} + \nu\omega_1 \sim 1 - \nu\frac{2}{9g} + O(\nu^2) \qquad (8.2) \\ \varepsilon^2 &= \nu^2\omega_1^2 + p\nu^2 + \mathcal{O}(\nu^3).. \qquad (8.3) \end{aligned}$$

Then from the definition of ν (2.16) it is clear that for $P \sim 2$,

$$\nu \sim \tfrac{3}{2}\left(\tfrac{38}{27} - a_3\right). \tag{8.4}$$

so that $\nu \to 0$ as $a_3 \to 38/27$. We calculate the value of p for comparison purposes. Using (8.1)-(8.3) we get the coefficients in the amplitude equation using (A.16)-(A.17)

$$\begin{aligned} d &= d_0 + d_1\nu = -32/27 + O(\nu) \qquad (8.5) \\ c &= c_0 + c_1\nu = 48.4\ldots + O(\nu) \qquad (8.6) \end{aligned}$$

which are used to determine the Maxwell load from (3.25). Using (3.25) and the definition of g (3.19)

$$\frac{\varepsilon^2}{\nu^2}c_0 \sim \frac{3}{4} + \frac{3d_0^2}{64} \tag{8.7}$$

We also use (3.25) to replace g in (8.2), and thus determine ε^2/ν^2 and p,

$$\begin{aligned} \frac{\varepsilon^2}{\nu^2} &= \frac{\varepsilon^4}{\nu^4}k + p, \quad k = \left(\frac{8}{27}\right)^2 \\ &\Rightarrow \quad p \approx \frac{\varepsilon^2}{\nu^2} \approx 0.017 \qquad (8.8) \end{aligned}$$

Then $2 - P_M$ is a quadratic function of $(38/27 - a_3)$,

$$P_M \sim 2 - \frac{9}{4}p(38/27 - a_3)^2. \tag{8.9}$$

This quadratic function is compared with the asymptotic result for P_M (3.25) in Figure 8.1.

We see that the result for P_M obtained from the quadratic approximation fails away from $P = 2$. The success of the asymptotic result of this paper is a direct result of an analysis

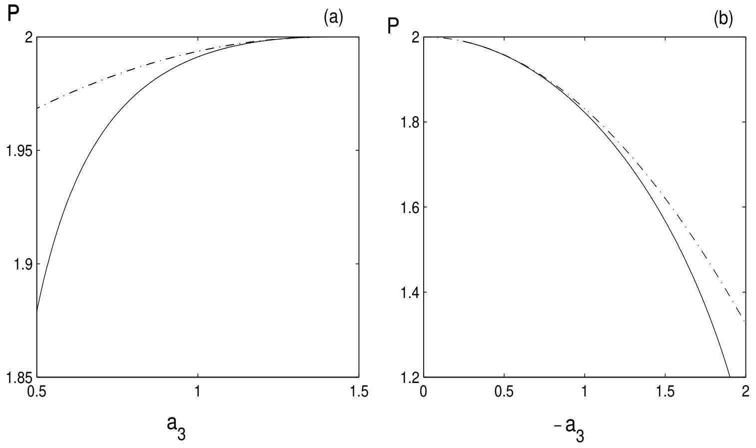

Figure 8.1: *Comparison of $2-P_M$ as a quadratic function of ν (dash-dotted line) with the asymptotic results from Sections 3 and 4 (solid line). a) Non-symmetric case $a_2=-1$, $a_4=a_5=0$, b) Symmetric case $a_3<0$, $a_5>0$, $a_2=a_4=0$.*

which is based on balancing the linear and nonlinear terms in the amplitude equation. This approach gives a more general formula for the Maxwell load (3.25) in which $2-P_M$ is not a quadratic function of ν. This result is valid for a range of values of P_M, such that $\varepsilon^2/\nu \ll 1$, as used in the derivation of the amplitude equation of Section 3.

We also compare to the symmetric case $a_3<0$, $a_5>0$, and $a_2=a_4=0$, which is equivalent to the analysis of wave number selection in the Swift-Hohenberg equation [3],[17] for P near 2. In this case $\nu^2=9a_3^2/(4a_5)$, and $\omega_1=0$, so that for P very close to 2 we find

$$P \sim 2-p\nu^2 \text{ for } \nu \ll 1. \tag{8.10}$$

$$\omega^2 \sim \frac{P}{2}+O(\nu^2) \tag{8.11}$$

$$\varepsilon^2 \sim p\nu^2. \tag{8.12}$$

Then the Maxwell condition is

$$a_5 p\nu^2 = a_3^2\left[\frac{27}{160}+\frac{3\nu^2 p}{640}\right] \quad \Rightarrow \quad p \sim \frac{3}{40} \tag{8.13}$$

$$\tag{8.14}$$

so that

$$P_M \approx 2-\frac{27a_3^2}{160a_5} = 2-\frac{3}{40}\nu^2.$$

In Figure 8.1 we compare this quadratic approximation with P_M obtained from (4.2).

9 Conclusions

The analysis presented in this paper allows us to consider the cellular solutions of the equation (2.1) for values of P in a significant range away from the critical bifurcation point. In particular we are able to analyse the transition between a homoclinic and a heteroclinic solution occuring at the Maxwell load P_M. The agreement between the numerical computations and the asymptotic calculations is excellent. The next stage in this analysis is to look at the asymptotic contribution of the beyond all orders terms close to P_M.

A Appendix: Coefficients in the amplitude equation

In Section 3 we showed that

$$u_4 = B_0(X) + B_2(X)e^{2i\omega x} + A_4(X)e^{4i\omega x} + \text{c.c.}$$

The coefficients $B_0(X)$ and $B_2(X)$ are given by

$$\begin{aligned}
\mu(2\omega)B_2 &= -a_2(2A_2A_0 + 2A_1^*A_3) - 4i\omega A_2'(X)(-8\omega^2 + P) - 3a_3(2|A_1|^2A_2 + A_1^2A_0) - 4a_4A_1^2|A_1|^2 \\
B_0 &= -a_2(2|A_2|^2 + A_0^2) - 3a_3(2|A_1|^2A_0 + A_1^2A_2^* + (A_1^*)^2A_2) - 6a_4|A_1|^4 \qquad \text{(A.15)}
\end{aligned}$$

The further coefficients c and d are given by

$$\begin{aligned}
c &= -\frac{12a_2^4}{\mu(2\omega)^2\mu(3\omega)} + \frac{12a_2^2a_3}{\mu(2\omega)\mu(3\omega)} - \frac{6}{\mu(2\omega)^2}(2a_2^4 - 3a_3a_2^2) \\
&\quad -4a_4\left(\frac{4a_2}{\mu(2\omega)} - 6a_2\right) + 10a_5 + \frac{36a_3a_2^2}{\mu(2\omega)} - \frac{3a_3^2}{\mu(3\omega)} - 4(2a_2^4 - 9a_3a_2^2) \qquad \text{(A.16)} \\
d &= -\frac{16a_2^2\omega(8\omega^2 - P)}{\mu(2\omega)^2} \qquad \text{(A.17)}
\end{aligned}$$

We note that in the definition of c the terms of the form $1/\mu(3\omega)$ and $1/\mu(2\omega)^2$ will be very small in comparison with the others when $\omega^2 = P/2$.

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