TWISTED SURFACES I: CLUSTERS OF CURVES

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Abstract. This paper explores the topological aspect of cluster theory.

Key words: quiver with potential, branched double cover, Dehn twist, spherical twist, cluster exchange groupoid

Introduction

Overall. This paper studies the topological aspect of cluster theory. Cluster algebra was invented by Fomin-Zelevinsky in 2000, which rapidly developed and have been shown related to various areas in mathematics (cf. [?]).

The combinatorial aspect of cluster theory is encoded by quiver mutation, which leads to the categorification by BMRRT via quiver representations. In the cluster category, mutation becomes tilting and it is an involution. Later, DWZ introduced the potential to quivers together with improved mutation. Keller-Yang gave the corresponding categorification of quivers with potential via the associated Ginzburg dg (=differential graded) algebra. In the corresponding Calabi-Yau-3 derived category, mutation becomes simple tilting and the square of mutation becomes Seidel-Thoams spherical twist.

The geometric aspect of cluster theory was explored by FST [8]. Let $\mathbf{S}$ be a marked surface. They showed that one can associated a quiver $Q_{\mathbf{T}}$ to each (tagged) triangulation $\mathbf{T}$ of $\mathbf{S}$ while the flip on triangulations becomes mutation on $Q_{\mathbf{T}}$. The remarkable feature in their story is the tagging which makes everything worked but also leaves a puzzle. Further, CL algebraically gave a potential for each $Q_{\mathbf{T}}$. The arcs in the marked surface corresponding to rigid objects in the corresponding cluster category, which play the role of projective.

Contents. Our main theorem is the following.

thm:0 **Theorem 0.1.** *Let $\mathbf{T}$ be a tagged triangulation on $\mathbf{S}$, $\mathbf{C_T}$ be the corresponding cluster of curves and $\mathfrak{C}_{\mathbf{T}}$ be canonical configuration of $\mathbf{C_T}$. Then*

- *the flip of an arc on $\mathbf{T}$ corresponds to the half Dehn twist of the corresponding curves on the cluster of curves $\mathbf{C_T}$ (Theorem **??**);*
- *the intersection quiver $Q(\mathbf{C_T})$ of $\mathbf{C_T}$ is isomorphic to the FST's quiver $Q_{\mathbf{T}}$ of $\mathbf{T}$ and the half Dehn twist on $\mathbf{C_T}$ induces mutation on $Q(\mathbf{C_T})$ (Theorem **??**);*
- *the embedded polygons of $\mathfrak{C}_{\mathbf{T}}$ induces a potential $W(\mathfrak{C}_{\mathbf{T}})$ for the intersection quiver $Q(\mathfrak{C}_{\mathbf{T}})$ of $\mathfrak{C}_{\mathbf{T}}$ such that the quiver with potential $(Q(\mathfrak{C}_{\mathbf{T}}), W(\mathfrak{C}_{\mathbf{T}}))$ is equivalent to CL's associated to $\mathbf{T}$ (Theorem **??**);*

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- *the reduced configuration of* $\mathfrak{C}_{\mathbf{T}}$ *induces the reduced quiver with potential of* $(Q(\mathfrak{C}_{\mathbf{T}}), W(\mathfrak{C}_{\mathbf{T}}))$ *such that the quiver is* $Q(\mathbf{C}_{\mathbf{T}})$*. (Theorem* **??***).*

Our main conjectures are the following

Conjecture 0.2. *Let* $\mathbf{T}$ *be a tagged triangulation on* $\mathbf{S}$*,* $\mathbf{C}_{\mathbf{T}}$ *be the corresponding cluster of curves and* $\mathfrak{C}_{\mathbf{T}}$ *be canonical configuration of* $\mathbf{C}_{\mathbf{T}}$ *on the twisted surface* $\Sigma_{\mathbf{T}}$*. Denote by* $(Q(\mathbf{C}_{\mathbf{T}}), W(\mathbf{C}_{\mathbf{T}}))$ *the reduced quiver with potential from* $\mathfrak{C}_{\mathbf{T}}$ *and* $\Gamma_{\mathbf{T}}$ *the assoicated Ginzburg dg algebra. Then*

- *there is a bijection between the set of reachable spherical objects in the finite-dimensional derived category* $\mathcal{D}(\Gamma_{\mathbf{T}})$ *of* $\Gamma_{\mathbf{T}}$ *quotient by shift* $[2]$ *and the set of reachable curves on* $\Sigma_{\mathbf{T}}$*;*
- *the total dimension of* $\mathrm{Hom}^{\bullet}$ *between any two of those spherical objects equals the geometry intersection number between the corresponding curves.*
- *the subgroup* $\mathrm{STG}(\Gamma_{\mathbf{T}})$*, generating by reachable spherical twists, of the auto-equivalence group* $\mathrm{Aut}\,\mathcal{D}(\Gamma_{\mathbf{T}})$ *is isomorphic to the subgroup* $\mathrm{DTG}(\Sigma_{\mathbf{T}})$*, generating by reachable Dehn twists, of the mapping class group* $\mathrm{MCG}(\Sigma_{\mathbf{T}})$*.*

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sec:IT

1. The Krammer groupoids

1.1. **Marked surfaces.** Fix be an algebraically close filed $\mathbf{k}$. $\mathbf{S}$ denotes an unpunctured *marked surface* in the sense of [8], that is, a connected Riemann surface with a fixed orientation with a finite set $\mathbf{M}$ of marked point on the boundary $\partial\mathbf{S}$ and a finite set $\mathbf{P}$ of punctures inside, satisfying the following conditions:

- $\mathbf{S}$ is not closed, i.e. $\partial\mathbf{S} \neq \emptyset$;
- each connected component of $\partial\mathbf{S}$ contains at least one marked point.

Up to homeomorphism, $\mathbf{S}$ is determined by the following data

- the genus g;
- the number b of boundary components;
- the number $p = \#\mathbf{P}$ of punctures;
- the integer partition of $m = \#\mathbf{M}$ into b parts describing the number of marked points on its boundary.

As in [8, p5], we will exclude the case when there is no triangulation or there is no arcs in the triangulation. In other wards, we require $n \geq 1$ in (1.1). In this paper, we have the following convention:

- an *open arc* is a curve on a marked surface that connected two marked points/punctures;
- a *curve* is a closed curve on a marked surface;
- any picture of $\mathbf{S}$ is drawn of its positive side.

An *ideal triangulation* T of $\mathbf{S}$ is a (isotopy class of) maximal collection of compatible open arcs. Here, compatibility means no intersection in $\mathbf{S} - \mathbf{M} - \mathbf{P}$. We have the following result of ideal triangulation.

pp:FST **Theorem 1.1** (cf. [8]). *For any ideal triangulation* T *of* $\mathbf{S}$, *it consists of*

$$n = 6g + 3p + 3b + m - 6 \tag{1.1}$$ eq:n

(simple essential) open arcs and divides $\mathbf{S}$ *in to* $\aleph = (2n+m)/3$ *triangles. Moreover, the unoriented exchange graph* $\underline{\mathrm{EG}}(\mathbf{S})$ *of ideal triangulations of* $\mathbf{S}$ *(whose edges correspond to the flip, cf. Figure 2) is connected with fundamental groups generated by squares and pentagons as in Figure 1.*

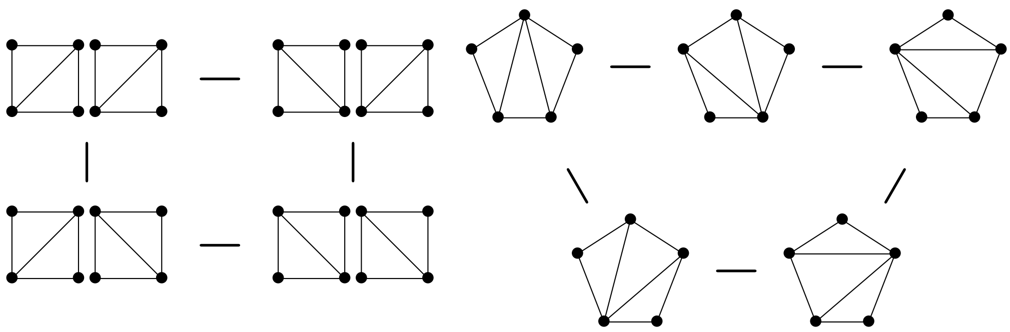


Figure 1. The square and pentagon fig:Pent

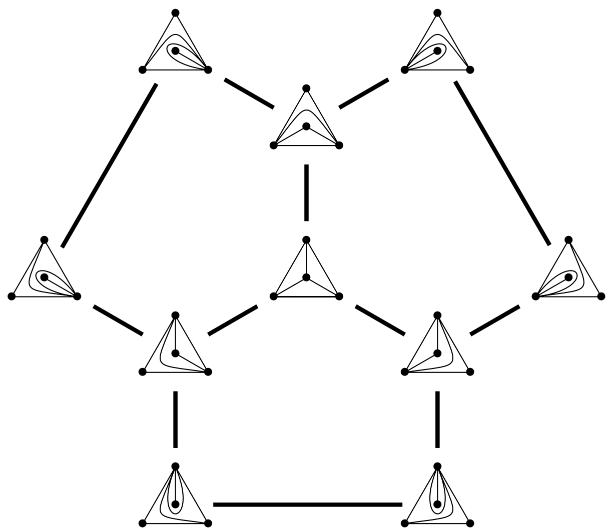


Figure 2. The (unoriented) exchange graph $\underline{\mathrm{EG}}(\mathbf{S})$ of ideal triangulations fig:FST

1.2. **The Krammer groupoids.** From now on, we will assume $\mathbf{S}$ is unpunctured, i.e. $p = 0$ within this section. There is the notion of (forward/backward) flip (after [19] and cf. [24]).

def:f.flip **Definition 1.2.** Let γ be an open arc in a triangulation T of $\mathbf{S}$. The arc $\gamma^\sharp = \gamma^\sharp(\mathrm{T})$ is the arc obtained from γ by anticlockwise moving its endpoints along the quadrilateral in T whose diagonal is γ to the next marked points. The *forward flip* of a triangulation T of $\mathbf{S}$ at $\gamma \in \mathrm{T}$ is the triangulation $\mathrm{T}^\sharp_\gamma$ obtained from T by replacing the arc γ with $\gamma^\sharp$.

Denote by $\mathrm{EG}^\circ(\mathbf{S})$ the *(oriented) exchange graph* of $\mathbf{S}$, whose vertices are ideal triangulations of $\mathbf{S}$ and whose edges are the forward flips.

Although the forward flip is the same as an ordinary flip, we will see later this will no longer be the case when we decorated $\mathbf{S}$. Moreover, an ordinary flip $\mathrm{T} \longrightarrow \mathrm{T}'$ in $\underline{\mathrm{EG}}(\mathbf{S})$ induces two forward flips $\mathrm{T} \longrightarrow \mathrm{T}'$ and $\mathrm{T} \longrightarrow \mathrm{T}'$ in $\mathrm{EG}^\circ(\mathbf{S})$, cf. Figure 3.

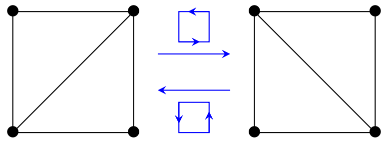


FIGURE 3. One ordinary flip induces two forward flips fig:F2-1

Definition 1.3. Let E be a oriented graph. Denote by $\mathcal{W}^+(\mathrm{E})$ the *path category* of E, i.e. whose objects are the vertices of E and whose generating morphisms corresponds the (oriented) edges of E. Denote by $\mathcal{W}(\mathrm{E})$ the *path groupoid* of E, i.e. the same presentation of $\mathcal{W}^+(\mathrm{E})$ but all the morphisms are invertible.

def:gpd **Definition 1.4.** Define the *Krammer groupoid* $\mathcal{EG}^\circ(\mathbf{S})$ of an (unpunctured) marked surface $\mathbf{S}$ to be the quotient groupoid of the path groupoid $\mathcal{W}(\mathrm{EG}^\circ(\mathbf{S}))$ by the following (square and pentagon) relations:

- For any square in $\mathrm{EG}^\circ(\mathbf{S})$ (cf. the left picture of Figure 1), it induces four oriented squares between local rotations as shown in Figure 4. We will impose the commutation relations in $\mathcal{EG}^\circ(\mathbf{S})$, that the compositions of generating morphisms along the two paths of any oriented square are equal.
- For any pentagon in $\mathrm{EG}^\circ(\mathbf{S})$ (cf. the right picture of Figure 1), it induces five oriented pentagon (each vertex of the original pentagon could be a source of such an oriented pentagon) between local rotations as shown in Figure 5. We will impose the pentagon relations in $\mathcal{EG}^\circ(\mathbf{S})$, that the compositions of generating morphisms along the two paths of any oriented square are equal.

1.3. **A covering via decorated surfaces.** Recall that any triangulation of $\mathbf{S}$ consists of $\aleph$ triangles.

def:arcs **Definition 1.5.** [24] The *decorated marked surface* $\mathbf{S}_\triangle$ is a marked surface $\mathbf{S}$ together with a fixed set $\triangle$ of $\aleph$ 'decorated' points (in the interior of $\mathbf{S}$. Moreover,

- An *open arc* in $\mathbf{S}_\triangle$ is (the isotopy class of) a simple curve in $\mathbf{S}_\triangle - \triangle$ that connects two marked points in $\mathbf{M}$.
- A *triangulation* $\mathbb{T}$ of $\mathbf{S}_\triangle$ is an (isotopy class of) maximal collection of compatible open arcs (i.e. no intersection in $\mathbf{S}-\mathbf{M}$) such that they divide $\mathbf{S}_\triangle$ into $\aleph$ triangles, each of which contains exactly one point in $\triangle$.

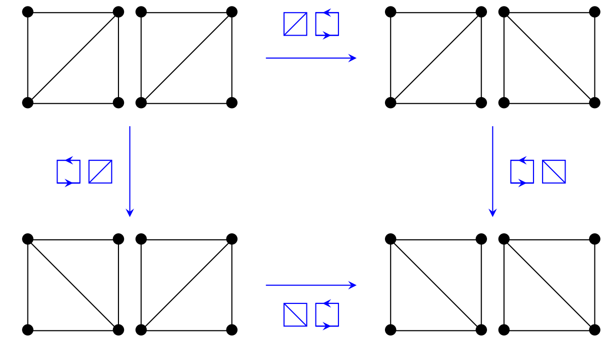


FIGURE 4. The commutation relation for $\mathcal{EG}^\circ(\mathbf{S})$ fig:Squa.rel

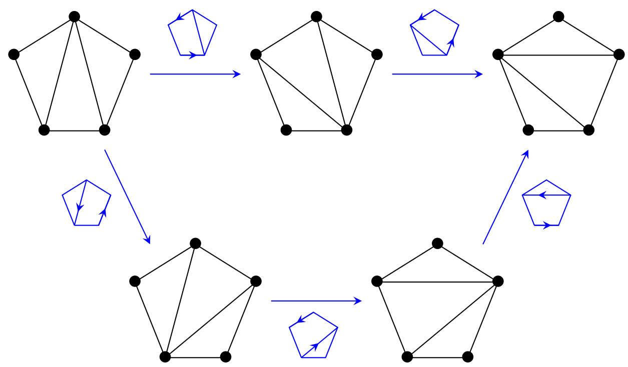


FIGURE 5. The pentagon relation for $\mathcal{EG}^\circ(\mathbf{S})$ fig:Pent.rel

- There is a canonical map, the *forgetful map*
$$F\colon \mathbf{S}_\triangle \to \mathbf{S},$$
forgetting the decorated points. Clearly, F induces a map from the set of open arcs in $\mathbf{S}_\triangle$ to the set of open arcs in $\mathbf{S}$. Thus, F also induces a map from the set of triangulations of $\mathbf{S}_\triangle$ to the set of triangulations of $\mathbf{S}$.
- The *forward flip* of a triangulation $\mathbb{T}$ of $\mathbf{S}_\triangle$ is defined exactly the same way as in Definition 1.2 for a triangulation T of $\mathbf{S}$. There relations are demonstrated in Figure 6.
- The *exchange graph* $\mathrm{EG}(\mathbf{S}_\triangle)$ is the oriented graph whose vertices are triangulations of $\mathbf{S}_\triangle$ and whose edges correspond to forward flips between them.

Note that $\mathrm{EG}(\mathbf{S}_\triangle)$ is usually not connected; however, each connected component are isomorphic to each other (cf. [24, Remark 3.10]). Now, we construct a covering of the groupoid $\mathcal{EG}^\circ(\mathbf{S})$. Fix an initial triangulation $\mathbb{T}_0$ of $\mathbf{S}_\triangle$ and take a connected component $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ of $\mathrm{EG}(\mathbf{S}_\triangle)$ that contains $\mathbb{T}_0$.

Denote by $\mathrm{T}_0 = F(\mathbb{T}_0)$ the induces triangulation of $\underline{\mathrm{EG}}(\mathbf{S})$.

def:gpd.b **Definition 1.6.** Define the *exchange groupoid* $\mathcal{EG}^\circ(\mathbf{S}_\triangle)$ to be the quotient groupoid of the path groupoid $\mathcal{W}(\mathrm{EG}^\circ(\mathbf{S}_\triangle))$ by the following (square and pentagon) relations:

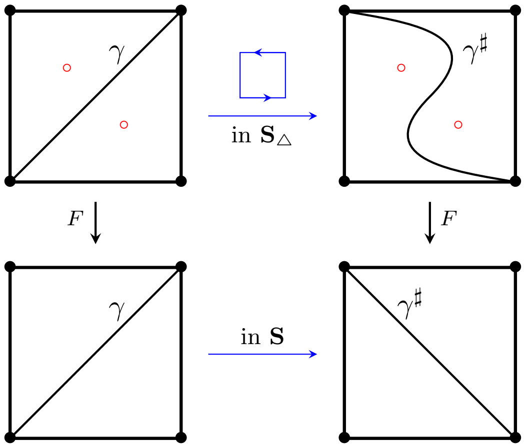


FIGURE 6. The forward flip

fig:filp.o

- For a triangulation $\mathbb{T}$ in $\mathrm{EG}^\circ(\mathbf{S})$ with two open arcs that are not adjacent in any triangle of $\mathbb{T}$ (cf. blue arcs in the upper left picture of Figure 7), the forward flips with respect to them form a square in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$. We will impose the corresponding pentagon relations in $\mathcal{EG}^\circ(\mathbf{S})$.
- For a triangulation $\mathbb{T}$ in $\mathrm{EG}^\circ(\mathbf{S})$ with two open arcs that are adjacent in some triangle of $\mathbb{T}$ (cf. blue arcs in the leftmost picture of Figure 21), it induces an oriented pentagon in $\mathrm{EG}^\circ(\mathbf{S}_\triangle)$ as shown in Figure 21. We will impose the corresponding pentagon relations in $\mathcal{EG}^\circ(\mathbf{S})$.

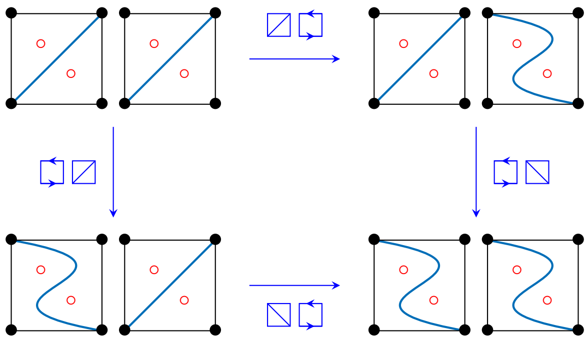


FIGURE 7. The commutation relation in $\mathcal{EG}^\circ(\mathbf{S}_\triangle)$

fig:4+

Lemma 1.7 (Krammer). *$\mathcal{EG}^\circ(\mathbf{S}_\triangle)$ is a covering of $\mathcal{EG}^\circ(\mathbf{S})$.*

Proof. We claim that the covering functor F_* is induced by the forgetful map F. It is clearly F_* is well-defined on the objects. Figure 6 show that F_* map the generating morphisms to the generating morphisms. What is left to check is that the relations between generating morphisms in $\mathcal{EG}^\circ(\mathbf{S})$ preserve by F_*. This follows from

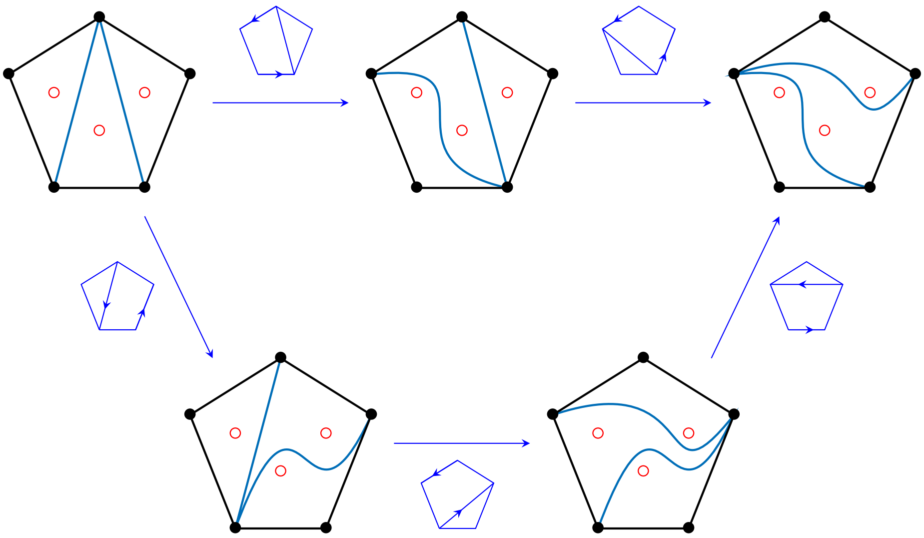


FIGURE 8. The pentagon relation for $\mathcal{EG}^\circ(\mathbf{S}_\triangle)$ fig:5+

- Figure 4 and Figure 5 for the square relations;
- Figure 7 and Figure 21 for the pentagon relations.

□

1.4. The covering group.

Definition 1.8. A *closed arc* in $\mathbf{S}_\triangle$ is (the isotopy class of) a simple curve in the interior of $\mathbf{S}_\triangle$ that connects two decorated points in $\triangle$. Let $\mathbb{T}$ be a triangulation of $\mathbf{S}_\triangle$ (consisting of n open arcs). The *dual triangulation* $\mathbb{T}^*$ of $\mathbb{T}$ is the collection of n closed arcs in $\mathrm{CA}(\mathbf{S}_\triangle)$, such that every closed arc only intersects one open arc in $\mathbb{T}$ and with intersection one. See the left picture of Figure 9 for an example. More precisely, for γ in $\mathbb{T}$, the corresponding closed arc in $\mathbb{T}^*$ is the unique open arc s, that is contained in the quadrilateral A with diagonal γ, connecting the two decorated points in A and intersecting γ only once We will call s and γ the dual of each other, with respect to $\mathbb{T}$ (or $\mathbb{T}^*$).

An *H-arc* in $\mathbf{S}_\triangle$ is (the isotopy class of) a simple curve in $\mathbf{S}_\triangle$ that connects a decorated point and a midpoint of some boundary segment of $\mathbf{S}_\triangle$. The completed dual triangulation of $\mathbb{T}$ consists of $\mathbb{T}^*$ and a maximal collection of H-arcs that does not intersect $\mathbb{T}$. So there will be exactly m H-arcs in any completed dual triangulation, cf. right picture of Figure 9.

Definition 1.9. The mapping class group $\mathrm{MCG}(\mathbf{S}_\triangle)$ is the group of isotopy classes of homeomorphisms of $\mathbf{S}_\triangle$, where all homeomorphisms and isotopies are required to

- fix $\partial\mathbf{S}_\triangle(\supset \mathbf{M})$ pointwise;
- fix the decorated points set $\triangle$ (but allow to permutate points in it).

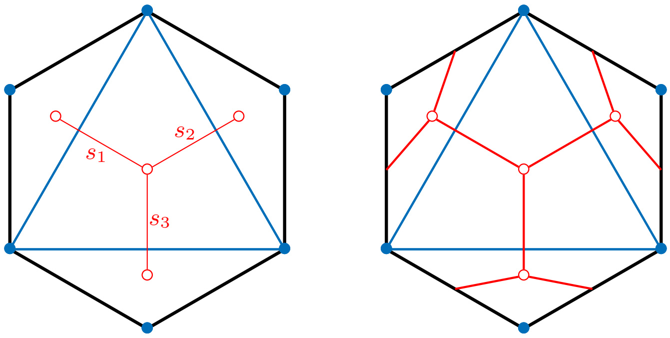


FIGURE 9. The (completed) dual of a triangulation

fig:ex0

Note that the mapping class group $\mathrm{MCG}(\mathbf{S})$ of $\mathbf{S}$ will require only the first condition. It is well-known that $\mathrm{MCG}(\mathbf{S})$ is generated by Dehn twists along simple closed curves.

For any closed arc $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$, there is the (positive) *braid twist* $\mathrm{B}_\eta \in \mathrm{MCG}(\mathbf{S}_\triangle)$ along η, which is shown in Figure 10. Further, there is the following well-known formula

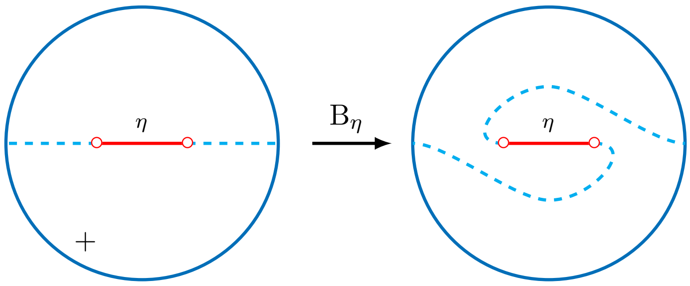


FIGURE 10. The Braid twist

fig:Braid twist

$$\mathrm{B}_{\rho(\eta)} = \rho \circ \mathrm{B}_\eta \circ \rho^{-1}, \tag{1.2}$$

eq:formulaB

for any $\Psi \in \mathrm{MCG}(\mathbf{S}_\triangle)$.

Definition 1.10. The *braid twist group* $\mathrm{BTG}(\mathbf{S}_\triangle)$ of the decorated marked surface $\mathbf{S}_\triangle$ is the subgroup of $\mathrm{MCG}(\mathbf{S}_\triangle)$ generated by the braid twists B_η for $\eta \in \mathrm{CA}(\mathbf{S}_\triangle)$.

Note that the composition of forward/backward flips is a negative/positive braid twist (cf. Figure 11). By [24, Lemma 3.9], we have the following.

lem:covering

Lemma 1.11. *The covering group of* $\mathcal{EG}^\circ(\mathbf{S}_\triangle) \to \mathcal{EG}^\circ(\mathbf{S})$ *is* $\mathrm{BTG}(\mathbf{S}_\triangle)$.

1.5. **The braid representation.** An alternative way to describe the covering in Lemma 1.11 is via a representation to the mapping class groupoid of decorated surfaces.

For any object (i.e. a triangulation) T in $\mathcal{EG}^\circ(\mathbf{S})$, define $\mathbf{S}_\triangle(\mathrm{T})$ to be the (isotopy class of) decorated marked surface obtained from $\mathbf{S}$ by decorating a set of $\aleph$ points so that each triangle of T contains exactly one decorated point. Note that there is a canonical triangulation $\mathbb{T}$ of $\mathbf{S}_\triangle(\mathrm{T})$, inherited from T.

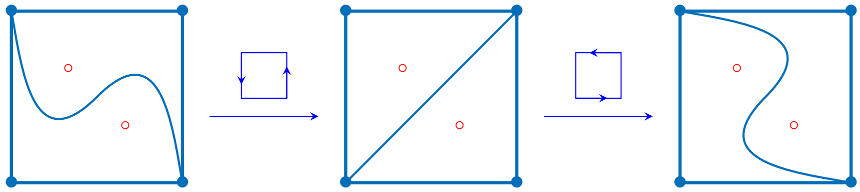


FIGURE 11. The composition of forward flips fig:flips

Definition 1.12. The *mapping class groupoid* $\mathcal{MCG}(\mathbf{S}_\triangle)$ of $\mathbf{S}_\triangle$ is the groupoid whose objects are $\mathbf{S}_\triangle(\mathrm{T})$ and whose morphisms are the (isotopy classed of) homeomorphisms between them.

Construction 1.13. Now, we define a representation $\xi_b\colon \mathcal{EG}^\circ(\mathbf{S}) \to \mathcal{MCG}(\mathbf{S}_\triangle)$ as follows:

- $\xi_b(\mathrm{T}) = \mathbf{S}_\triangle(\mathrm{T})$ for any object (i.e. a triangulation) T in $\mathcal{EG}^\circ(\mathbf{S})$;
- For each generating morphism (i.e. a forward flip) $\eta\colon \mathrm{T} \to \mathrm{T}' = \mathrm{T}^\sharp_\gamma$, let $\xi_b(\eta)$ be the (isotopy class of) homeomorphism from $\mathbf{S}_\triangle(\mathrm{T})$ to $\mathbf{S}_\triangle(\mathrm{T}^\sharp_\gamma)$, such that
$$\xi(\eta)\left(\mathbb{T}^\sharp_\gamma\right) = \mathbb{T}', \tag{1.3}$$
eq:def
where $\mathbb{T}$ and $\mathbb{T}'$ are the canonical triangulations of $\mathbf{S}_\triangle(\mathrm{T})$ and $\mathbf{S}_\triangle(\mathrm{T}')$, respectively.

Note that triangulations of decorated surfaces satisfy the square and pentagon relations (cf. Figure 7 and Figure 21). Similarly, by Figure **??**, there are square and pentagon relations in $\mathcal{MCG}(\mathbf{S}_\triangle)$ by (1.4). Then it is straightforward to check the representation above is well-defined. Moreover, denoted by $\mathcal{BT}(\mathbf{S}_\triangle)$ the image of ξ_b. Then Lemma 1.11 implies
$$\pi_1(\mathcal{BT}(\mathbf{S}_\triangle)) \cong \mathrm{BTG}(\mathbf{S}_\triangle). \tag{1.4}$$
eq:def

2. Quivers with potential and cluster theory

In this section, we discuss the cluster exchange groupoids, which is a generalization of the Krammer groupoids.

2.1. **Three categories.** A quiver Q is a directed graph and a potential W associated to Q is the sum of some cycles in Q (possible with coefficients). One can mutate a quiver with potential, in the sense of DWZ. We will always assume the quivers with potential are *non-degenerated*, which basically means that the (iterated) mutation works well. Further details see [14] for example.

Denote by $\Gamma(Q,W)$ the *Ginzburg dg algebra (of degree 3)* associated to a quiver with potential (Q,W), which is constructed as follows (see [14, § 7.2] for further details):

- Let Q^3 be the graded quiver whose vertex set is Q_0 and whose arrows are: the arrows in Q with degree 0; an arrow $a^* : j \to i$ with degree -1 for each arrow $a : i \to j$ in Q; a loop $e^* : i \to i$ with degree -2 for each vertex e in Q.

- The underlying graded algebra of $\Gamma(Q,W)$ is the completion of the graded path algebra $\mathbf{k}Q^3$ in the category of graded vector spaces with respect to the ideal generated by the arrow of Q^3.
- The differential of $\Gamma(Q,W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule and takes the following values
$$\mathrm{d}\sum_{e\in Q_0} e^* + \mathrm{d}\sum_{a\in Q_1} a^* = \sum_{a\in Q_1}[a,a^*] + \partial W.$$

A triangulated category $\mathcal{D}$ is called *N-Calabi-Yau* (N-CY) if, for any objects L,M in $\mathcal{D}$ we have a natural isomorphism

$$\mathfrak{S}: \mathrm{Hom}^\bullet_{\mathcal{D}}(L,M) \xrightarrow{\sim} \mathrm{Hom}^\bullet_{\mathcal{D}}(M,L)^\vee[N]. \tag{2.1}$$ eq:serre

Further, an object S is *N-spherical* if $\mathrm{Hom}^\bullet(S,S) = \mathbf{k}\oplus\mathbf{k}[-N]$ and (2.1) holds functorially for $L=S$ and M in $\mathcal{D}$.. Note that the graded dual of a graded $\mathbf{k}$-vector space $V=\oplus_{i\in\mathbb{Z}}V_i[i]$ is
$$V^\vee = \bigoplus_{i\in\mathbb{Z}} V_i^*[-i].$$
There are three categories associated to a Ginzburg dg algebra Γ, namely,

- The perfect derived category $\operatorname{per}\Gamma$.
- The finite-dimensional derived category $\mathcal{D}_{fd}(\Gamma)$, which is a full subcategory of $\operatorname{per}\Gamma$. Moreover, it is 3-CY.
- The cluster category $\mathcal{C}(\Gamma)$, which is defined by the following short exact sequence of triangulated categories (due to Amiot)
$$0 \to D_{fd}(\Gamma) \to \operatorname{per}\Gamma \to \mathcal{C}(\Gamma) \to 0. \tag{2.2}$$ eq:ses

  Moreover, it is 2-CY.

ex:un-p **Example 2.1.** Let $\mathbf{S}$ be an unpunctured marked surface with a triangulation T. Then there is an associated FST-quiver Q_{T} with a LF-potential W_{T}, constructed as follows (See, e.g. [10] or [27] for the precise definition):

- the vertices of Q_{T} are the arcs in T.
- for each triangle T (a.k.a. the type I puzzle piece) in T, there are three arrows between the corresponding vertices as shown in Figure 12.
- these three arrows form a 3-cycle in Q_{T} and W_{T} is the sum of all such 3-cycles that correspond to inner triangles (i.e. whose edges, say b,c,d in Figure 12, are not boundary segments).

Denote by the corresponding Ginzburg dg algebra of such a quiver with potential by Γ_{T}.

2.2. **Cluster tilting and cluster exchange graphs.** A *cluster tilting set* $\mathbf{L}$ in a category $\mathcal{C}$ is an Ext-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\mathrm{Hom}^1(L,M)=0$ for any $L,M\in\mathbf{L}$. The *forward mutation* $\mu_L^\sharp$ at an element $L\in\mathbf{L}$ acts on a cluster tilting set $\mathbf{L}$ by replacing L by

$$L^\sharp = \mathrm{Cone}\Big(L \to \bigoplus_{M\in\mathbf{L}-\{L\}} \mathrm{Irr}(L,M)^*\otimes T\Big), \tag{2.3}$$ eq:++

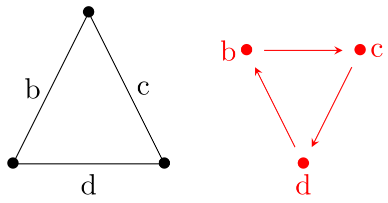


FIGURE 12. The type I puzzle piece and associated quiver with potential fig:type I

where $\mathrm{Irr}(X, Y)$ is a space of irreducible maps $X \to Y$, in the additive subcategory $\mathrm{Add} \bigoplus_{M \in \mathbf{L}} M$ of $\mathcal{C}$. The *backward mutation* $\mu_L^\flat$ at an element $L \in \mathbf{L}$ acts on a cluster tilting set $\mathbf{L}$ by replacing L by

$$L^\flat = \mathrm{Cone}\Big(\bigoplus_{M \in \mathbf{L} - \{L\}} \mathrm{Irr}(M, L) \otimes M \to L \Big)[-1]. \tag{2.4}$$ eq:--

If $\mathcal{C}$ is 2-CY, we will have $L^\sharp = L^\flat$ or $\mu_L^\sharp \mathbf{L} = \mu_L^\flat \mathbf{L}$.

For a cluster category $\mathcal{C}(\Gamma)$ arising from a Ginzburg dg algebra Γ of some quiver with potential (Q, W), there is a canonical cluster tilting set $\mathbf{L}_\Gamma$ whose elements correspond one-one to the vertices of the quiver Q. It is in fact induced from the canonical *silting object* Γ in $\mathrm{per}\,\Gamma$, cf. § 2.5. All the cluster tilting sets that can be iterated mutated from $\mathbf{L}_\Gamma$ are called *reachable*. Note that if (Q, W) is rising from a marked surface (with non-empty boundaries and possibly with punctures), then every cluster tilting set is reachable ([27, Theorem 5.2]).

Definition 2.2. Let Γ be the Ginzburg dg algebra of some quiver with potential. The *(oriented) cluster exchange graph* $\mathrm{CEG}(\mathcal{C}(\Gamma))$ of the cluster category $\mathcal{C}(\Gamma)$ is the oriented graph whose vertices are the reachable cluster tilting sets in $\mathcal{C}(\Gamma)$ and whose edges are the forward mutations.

Let $\mathfrak{Q}$ be a mutation equivalent class of (non-degenerated) quivers with potential. There is notation of *(unoriented) cluster exchange graph* $\underline{\mathrm{CEG}}(\mathfrak{Q})$ of $\mathfrak{Q}$, whose vertices are quivers with potential and whose edges are mutations between them. One can consider its oriented version $\mathrm{CEG}(\mathfrak{Q})$ by replacing every edge with a 2-cycle (i.e. give the mutation a direction, say forward/backward), cf [16, Figure 4].

Remark 2.3. For each cluster tilting set $\mathbf{L}$ in $\mathrm{CEG}(\mathcal{C}(\Gamma))$, one can associated a quiver with potential $(Q_\mathbf{L}, W_\mathbf{L})$ so that the quiver $Q_\mathbf{L}$ is the Gabriel quiver for $\mathbf{L}$. It is well-known that $\mathrm{CEG}(\mathfrak{Q})$ can be identified (as graphs) with $\mathrm{CEG}(\mathcal{C}(\Gamma(Q, W)))$ for any (Q, W) in $\mathfrak{Q}$, where the correspondence is given by

$$\begin{array}{ccc} \mathrm{CEG}(\mathcal{C}(\Gamma(Q, W))) & \cong & \mathrm{CEG}(\mathfrak{Q}) \\ \mathbf{L} & \mapsto & (Q_\mathbf{L}, W_\mathbf{L}). \end{array}$$

So the (forward/backward) mutation between cluster titling sets becomes the (forward/backward) DWZ-mutation for the associated quivers with potential.

2.3. **Exchange graphs of hearts.** There is a close relation between cluster exchange graphs and exchange graphs of hearts in the corresponding 3-CY categories $\mathcal{D}_{fd}(\Gamma)$. We will investigate this in this subsection.

A *t-structure* $\mathcal{P}$ on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{P} \subset \mathcal{D}$ with $\mathcal{P}[1] \subset \mathcal{P}$ and such that, if one defines

$$\mathcal{P}^{\perp} = \{G \in \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}(F, G) = 0, \forall F \in \mathcal{P}\},$$

then, for every object $E \in \mathcal{D}$, there is a unique triangle $F \to E \to G \to F[1]$ in $\mathcal{D}$ with $F \in \mathcal{P}$ and $G \in \mathcal{P}^{\perp}$. It is *bounded* if for any object M in $\mathcal{D}$, the shifts $M[k]$ are in $\mathcal{P}$ for $k \gg 0$ and in $\mathcal{P}^{\perp}$ for $k \ll 0$. We will only consider bounded t-structures. The *heart* of a (bounded) t-structure $\mathcal{P}$ is the full subcategory $\mathcal{H} = \mathcal{P}^{\perp}[1] \cap \mathcal{P}$, which uniquely determines $\mathcal{P}$.

Note that any heart is abelian. Recall that a *torsion pair* in an abelian category $\mathcal{A}$ is a pair of full subcategories $\langle \mathcal{F}, \mathcal{T} \rangle$ of $\mathcal{A}$, such that $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and furthermore every object $E \in \mathcal{A}$ fits into a short exact sequence $0 \longrightarrow E^{\mathcal{T}} \longrightarrow E \longrightarrow E^{\mathcal{F}} \longrightarrow 0$ for some objects $E^{\mathcal{T}} \in \mathcal{T}$ and $E^{\mathcal{F}} \in \mathcal{F}$. We will write $\mathcal{A} = \langle \mathcal{F}, \mathcal{T} \rangle$.

Let $\mathcal{H}$ be a heart in a triangulated category $\mathcal{D}$ with torsion pair $\mathcal{H} = \langle \mathcal{F}, \mathcal{T} \rangle$. Then there is a heart $\mathcal{H}^{\sharp}$ with torsion pair $\mathcal{H}^{\sharp} = \langle \mathcal{T}, \mathcal{F}[1] \rangle$, called the *forward tilts* of $\mathcal{H}$ and a heart $\mathcal{H}^{\flat}$ with torsion pair $\mathcal{H}^{\flat} = \langle \mathcal{T}[-1], \mathcal{F} \rangle$, called the *backward tilts* of $\mathcal{H}$ (cf. [16, Proposition 3.2]). We say a forward tilt is *simple* if $\mathcal{F} = \langle S \rangle$ for a rigid simple S, and write it as $\mathcal{H}^{\sharp}_{S}$. Similarly, a backward tilt is *simple* if $\mathcal{T} = \langle S \rangle$ for a rigid simple S, and write it as $\mathcal{H}^{\flat}_{S}$.

Let $\Gamma = \Gamma(Q, W)$ be the Ginzburg dg algebra of some quiver with potential (Q, W). Then $\mathcal{D}_{fd}(\Gamma)$ admits a canonical heart $\mathcal{H}_{\Gamma}$ generated by simple Γ-modules S_e, for $e \in Q_0$, each of which is 3-spherical, i.e. (cf. say [14]).

Definition 2.4. The *twist functor* ϕ *of a spherical object* S is defined by

$$\phi_S(X) = \mathrm{Cone}\,(S \otimes \mathrm{Hom}^{\bullet}(S, X) \to X) \tag{2.5}$$

eq:phi

with inverse

$$\phi_S^{-1}(X) = \mathrm{Cone}\left(X \to S \otimes \mathrm{Hom}^{\bullet}(X, S)^{\vee}\right)[-1].$$

Denote by $\mathrm{STG}(\Gamma)$ the spherical twist group of $\mathcal{D}_{fd}(\Gamma)$ in $\mathrm{Aut}\,\mathcal{D}_{fd}(\Gamma)$, generated by $\{\phi_{S_e} \mid e \in Q_0\}$.

The *(total) exchange graph* $\mathrm{EG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the oriented graph whose vertices are all hearts in $\mathcal{D}$ and whose edges correspond to simple forward tiltings between them. We will consider the principal component $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ of the exchange graph $\mathrm{EG}(\mathcal{D}_{fd}(\Gamma))$ of $\mathcal{D}_{fd}(\Gamma)$. We call the hearts in $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ *reachable*.

Denote by $\mathrm{Sph}(\Gamma)$ the set of *reachable* spherical objects in $\mathcal{D}_{fd}(\Gamma)$, that is,

$$\mathrm{Sph}(\Gamma) = \mathrm{STG}(\Gamma) \cdot \mathrm{Sim}\,\mathcal{H}_{\Gamma}, \tag{2.6}$$

eq:sph=st

where $\mathrm{Sim}\,\mathcal{H}$ denotes the set of simples of an abelian category $\mathcal{H}$. Then $\mathrm{Sph}(\Gamma)$ in fact consists of all the simples of *reachable* hearts (see, e.g. [24]).

We have the following result (cf. [16, Theorem 8.6] for the acyclic case).

thm:KN **Theorem 2.5.** [13, Theorem 5.6] *Let Γ be the Ginzburg dg algebra of some quiver with potential. There is an isomorphism between oriented graphs:*

$$\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))/\operatorname{STG}(\Gamma)=\operatorname{CEG}(\mathcal{C}(\Gamma)). \tag{2.7}$$ eq:quo

Moreover, let $\mathcal{H}$ be a heart in $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ that corresponds to a cluster tilting set $\mathbf{L}$ in $\operatorname{CEG}(\mathcal{C}(\Gamma))$. Then the associated/Gabriel quiver $Q_{\mathbf{L}}$ can be identified with the degree one part of the Ext-quiver *of $\mathcal{H}$ (cf. [16, § 6]).*

eq:pentagon

2.4. **Square and pentagon relations.** In this section, we show that $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ has squares and pentagons.

Let $\mathcal{D}$ be a triangulated category with hearts $\mathcal{H}_1\le\mathcal{H}_2\le\mathcal{H}_3\le\mathcal{H}_1[1]$. By [16, Proposition 3.2], there is a torsion pair $\langle\mathcal{F}_{ij},\mathcal{T}_{ij}\rangle$ in $\mathcal{H}_i$ such that $\mathcal{H}_j$ is the forward tilting of $\mathcal{H}_i$ with respect to which, for $(i,j)=(1,2),(2,3),(1,3)$. In fact, we have

$$\mathcal{T}_{ij}=\mathcal{H}_i\cap\mathcal{H}_j,\quad \mathcal{F}_{ij}=\mathcal{H}_j[-1]\cap\mathcal{H}_i\,. \tag{2.8}$$ eq:FT

lem:123 **Lemma 2.6.** *$\mathcal{F}_{13}$ admits a torsion pair $\langle\mathcal{F}_{12},\mathcal{F}_{23}\rangle$ in the sense that*

- *1°. $\operatorname{Hom}_{\mathcal{D}}(\mathcal{F}_{23},\mathcal{F}_{12})=0$;*
- *2°. for any object X in $\mathcal{F}_{13}$, there is a short exact sequence $0\to N\to X\to L\to 0$ in $\mathcal{H}_1$, where $N\in\mathcal{F}_{23},L\in\mathcal{F}_{12}$;*
- *3°. If there is a short exact sequence $0\to N\to X\to L\to 0$ in $\mathcal{H}_1$, where $N\in\mathcal{F}_{23},L\in\mathcal{F}_{12}$, then X is in $\mathcal{F}_{13}$.*

Proof. Recall that a heart in a triangulated category induces a homology (cf. [16, (2.4)]). Denote by $\mathbf{H}_\bullet(?)$ the homology with respect to $\mathcal{H}_1$. Since $\mathcal{H}_1\le\mathcal{H}_2\le\mathcal{H}_3\le\mathcal{H}_1[1]$, $\mathbf{H}_\bullet(X)$ concentrates in degree zero and one, for any objects X in $\mathcal{H}_2$ or $\mathcal{H}_3$. Since $\mathcal{F}_{23}=\mathcal{H}_2\cap\mathcal{H}_3[-1]$, $\mathbf{H}_\bullet(X)$ must concentrate in degree zero, for any $X\in\mathcal{F}_{23}$. In other words, $\mathcal{F}_{23}\in\mathcal{H}_1$. Then by (2.8) we have

$$\begin{aligned}\mathcal{T}_{12}\cap\mathcal{F}_{13} &= (\mathcal{H}_1\cap\mathcal{H}_2)\cap(\mathcal{H}_3[-1]\cap\mathcal{H}_1)\\ &= \mathcal{H}_1\cap(\mathcal{H}_3[-1]\cap\mathcal{H}_2)\\ &= \mathcal{H}_1\cap\mathcal{F}_{23}=\mathcal{F}_{23}.\end{aligned}$$

Similarly, we have $\mathcal{T}_{23}[-1]\cap\mathcal{F}_{13}=\mathcal{F}_{12}$. So 1° follows from $\operatorname{Hom}_{\mathcal{D}}(\mathcal{T}_{12},\mathcal{F}_{12})=0$ and $\mathcal{F}_{23}\subset\mathcal{T}_{12}$. For 2°, such an X admits a short exact sequence $0\to N\to X\to L\to 0$ in $\mathcal{H}_1$, where $N\in\mathcal{T}_{12},L\in\mathcal{F}_{12}$. Since $\mathcal{F}_{23}$ is closed under taking subobject, we have N is in $\mathcal{F}_{23}$ and hence in $\mathcal{T}_{12}\cap\mathcal{F}_{13}=\mathcal{F}_{23}$ as required. To finish, let N,X,L satisfy the condition in 3°. Then $\mathcal{F}_{23}\in\mathcal{T}_{12}$ implies $X\in\mathcal{H}_1$ and $\mathcal{F}_{12}\in\mathcal{T}_{23}[-1]$ implies $X\in\mathcal{H}_3[-1]$. Thus $X\in\mathcal{H}_3[-1]\cap\mathcal{H}_1=\mathcal{F}_{13}$. □

pp:eg.h **Proposition 2.7.** *Let $\mathcal{H}$ be a heart in $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ with simples S_i and S_j satisfying* $\operatorname{Ext}^1(S_i,S_j)=0$. *We have the following.*

- *(I). If $\operatorname{Hom}^1(S_j,S_i)=0$ and then $(\mathcal{H}_i)^\sharp_{S_j}=\mathcal{H}_{ij}$. Equivalently, there is a square in $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ as in the left diagram of (2.9).*
- *(II). If $\operatorname{Hom}^1(S_j,S_i)=\mathbf{k}$ and then $\mathcal{H}_{ij}=(\mathcal{H}_*)^\sharp_{S_j}$, where $T_j=\phi^{-1}_{S_i}(S_j)$ and $\mathcal{H}_*=(\mathcal{H}_i)^\sharp_{T_j}$. Equivalently, there is a pentagon in $\mathrm{EG}^{\circ}(\mathcal{D}_{fd}(\Gamma))$ as in the right diagram of (2.9).*

$$
\begin{tikzcd}
 & \mathcal{H}_i \arrow[dr, "S_j"] & \\
\mathcal{H} \arrow[ur, "S_i"] \arrow[dr, "S_j"'] & & \mathcal{H}_{ij} \\
 & \mathcal{H}_j \arrow[ur, "S_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{H}_i \arrow[r, "T_j"] & \mathcal{H}_* \arrow[dr, "S_j"] & \\
\mathcal{H} \arrow[ur, "S_i"] \arrow[drr, "S_j"'] & & & \mathcal{H}_{ij} \\
 & & \mathcal{H}_j \arrow[ur, "S_i"'] &
\end{tikzcd}
\tag{2.9}
$$ eq:45

Proof. We only deal the pentagon case since the square case is similar and simpler. Let $\mathcal{F}$ be the abelian category generated by S_i and S_j, which contains only three indecomposables, namely S_i, S_j and T_j. By [16, Proposition 5.2], we know that S_i is a simple in $\mathcal{H}_j$. Moreover, $\mathcal{H}_{ji} \leq \mathcal{H}[1]$ by [16, Lemma 5.4]. Applying Lemma 2.6, we see that $\mathcal{H}_{ij}$ is the forward tilt of $\mathcal{H}$ with respect to the torsion pair whose torsion free part is $\mathcal{F}$.

Similarly, T_j is a simple in $\mathcal{H}_i$ and S_j is a simple in $\mathcal{H}_*$. Let $\mathcal{H}_{ji} = (\mathcal{H}_*)^{\sharp}_{S_j}$ and we have $\mathcal{H}_{ji} \leq \mathcal{H}[1]$ by [16, Lemma 5.4]. Applying Lemma 2.6, we see that $\mathcal{H}_*$ is the forward tilt of $\mathcal{H}$ with respect to the torsion pair whose torsion free part is $\mathcal{F}'$, which contains exactly two indecomposables S_i and T_j. Applying Lemma 2.6 again, we see that $\mathcal{H}_{ji}$ is the forward tilt of $\mathcal{H}$ with respect to the torsion pair whose torsion free part contains exactly three indecomposables, S_i T_j and S_j, which is exactly $\mathcal{F}$. Thus $\mathcal{H}_{ij} = \mathcal{H}_{ji}$, i.e. we have the pentagon in (2.9) as required. $\square$

def:gpd.h **Definition 2.8.** Define the *exchange groupoid* $\mathcal{EG}^\circ(\mathcal{D}_{fd}(\Gamma))$ to be the quotient groupoid of the path groupoid $\mathcal{W}(\mathcal{EG}^\circ(\mathcal{D}_{fd}(\Gamma)))$ by the square and pentagon relations (induced from squares/pentagons in (2.9)) as in Definition 1.6.

By Theorem 2.5, squares and pentagons in (2.9) induces squares and pentagons in $\mathrm{CEG}(\mathcal{C}(\Gamma))$ in the following sense.

Lemma 2.9. *Let* $\mathbf{L}$ *be a cluster tilting set. Let* $L_i, L_j \in \mathbf{L}$ *such that there is no arrow from* L_i *to* L_j *in the Gabriel quiver* $Q_{\mathbf{L}}$. *We have the following.*

- *(I). If there is no arrow from* L_j *to* L_i *in* $Q_{\mathbf{L}}$, *then there is a square in* $\mathrm{CEG}(\mathcal{C}(\Gamma))$ *as in the left diagram of* (2.9).
- *(II). If there is exactly one arrow from* L_j *to* L_i *in* $Q_{\mathbf{L}}$, *then there is a pentagon in* $\mathrm{CEG}(\mathcal{C}(\Gamma))$ *as in the right diagram of* (2.9).

$$
\begin{tikzcd}
 & \mathbf{L}_i \arrow[dr, "L_j"] & \\
\mathbf{L} \arrow[ur, "L_i"] \arrow[dr, "L_j"'] & & \mathbf{L}_{ij} \\
 & \mathbf{L}_j \arrow[ur, "L_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbf{L}_i \arrow[r, "L_j"] & \mathbf{L}_* \arrow[dr, "L_i'"] & \\
\mathbf{L} \arrow[ur, "L_i"] \arrow[drr, "L_j"'] & & & \mathbf{L}_{ij} \\
 & & \mathbf{L}_j \arrow[ur, "L_i"'] &
\end{tikzcd}
\tag{2.10}
$$ eq:45.2

Proof. $\square$

def:gpd.c **Definition 2.10.** Let (Q, W) be a quiver with potential in a mutation equivalent class $\mathfrak{Q}$ and $\Gamma = \Gamma(Q, W)$ the Ginzburg dg algebra. Define the *cluster exchange groupoid* $\mathcal{CEG}(\Gamma)$ to be the quotient groupoid of the path groupoid $\mathcal{W}(\mathrm{CEG}(\mathcal{C}(\Gamma)))$ by the square and pentagon relations (induced from squares/pentagons in (2.10)) as in Definition 1.6.

By construction, we have the following covering

$$\mathcal{EG}^\circ(\mathcal{D}_{fd}(\Gamma) \twoheadrightarrow \mathcal{CEG}(\Gamma). \tag{2.11}$$ eq:cov

The covering group is the $\mathrm{STG}_0(\Gamma)$, which is a quotient group of $\mathrm{STG}(\Gamma)$ that acts freely on $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma)$ (or equivalently on the corresponding component of stability space, cf. [26]). As a reachable heart is finite and determined by its simples, the condition that acts freely on $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma)$ is equivalent to acts freely on $\mathrm{Sph}(\Gamma)$.

sec:cat

2.5. **Silting mutation and derived equivalences.** In the following two subsections, we give a categorical representation of cluster exchange groupoid, which is a upgraded version of (2.11).

A *silting object* $\mathbf{M}$ in a category $\mathcal{C}$ is an object such that it generates $\mathcal{C}$ (i.e. $\mathrm{thick}(\mathbf{M}) = \mathcal{C}$) and any two indecomposable summands M, M' of $\mathbf{M}$ satisfy $\mathrm{Hom}^t(M, M') = 0$ if $t > 0$. We will also usually requires that the indecomposable summands of $\mathbf{M}$ are pairwise non-isomorphic. For instance, Γ is the canonical silting object in $\mathrm{per}\,\Gamma$ for a Ginzburg dg algebra of some quiver with potential. Using the formulae (2.3) and (2.3) for cluster tilting, one can define the forward/backward mutation of a silting object, by replacing an indecomposable summand with another one. As before, we will call a silting object in $\mathrm{per}\,\Gamma$ reachable, if it can be iterated mutated from Γ. Denote the *exchange graph* of reachable silting objects in $\mathrm{per}\,\Gamma$ by $\mathrm{EG}^\circ(\mathrm{per}\,\Gamma)$. By Keller-Nicolás (cf. [18]), we have the following.

pp:dual

Proposition 2.11. *There is a canonical isomorphism between oriented graphs*

$$\mathrm{EG}^\circ(\mathrm{per}\,\Gamma) \cong \mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma)),$$

which is a dg version of projective-simple duality. More precisely, if $\mathbf{M} = \bigcup_{i=1}^n M_i$ *in* $\mathrm{EG}^\circ(\mathrm{per}\,\Gamma)$ *corresponds to* $\mathcal{H}$ *in* $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma))$*, then one can label the simples of* $\mathcal{H}$ *as* $\{S_i\}_{i=1}^n$ *so that* $\mathrm{Hom}^\bullet(M_i, S_j) = \delta_{ij}\mathbf{k}$.

In fact, the proof of (2.7) uses the proposition above: given a heart $\mathcal{H}$ in $\mathrm{EG}^\circ(\mathcal{D}_{fd}(\Gamma))$, let $\mathbf{M}$ be the corresponding silting object in $\mathrm{per}\,\Gamma$; then its image in $\mathcal{C}(\Gamma)$ under the projection in (2.2) will be a cluster tilting set.

Beside, (2.9) in Proposition 2.7 induces squares and pentagons in $\mathrm{EG}^\circ(\mathrm{per}\,\Gamma)$ as follows:

$$\begin{tikzcd}[column sep=small] & \mathbf{X}_i \arrow[dr, "\mu_j"] & \\ \mathbf{X} \arrow[ur, "\mu_i"] \arrow[dr, "\mu_j"'] & & \mathbf{X}_{ij} \\ & \mathbf{X}_j \arrow[ur, "\mu_i"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathbf{X}_i \arrow[rr, "\mu_j"] & & \mathbf{X}_* \arrow[dr, "\mu_i'"] & \\ \mathbf{X} \arrow[ur, "\mu_i"] \arrow[drr, "\mu_j"'] & & & & \mathbf{X}_{ij} \\ & & \mathbf{X}_j \arrow[urr, "\mu_i"'] & & \end{tikzcd} \tag{2.12}$$ eq:45.3

sec:cat2

2.6. **A categorical representation.** Next, we study the categorical version of the mutation of silting objects in $\mathrm{per}\,\Gamma$. Let (Q, W) be a quiver with potential and $(Q', W') = \mu_i(Q, W)$ be the quiver with potential mutated from (Q, W) at vertex i. Denote by Γ and Γ' by their Ginzburg dg algebras respectively. Let S_i and S_i' be the simples corresponding to i in the canonical hearts $\mathcal{H}_\Gamma$ and $\mathcal{H}_\Gamma'$ of $\mathcal{D}_{fd}(\Gamma)$ and $\mathcal{D}_{fd}(\Gamma')$ respectively.

Let $\mathbf{X}$ be the silting objects in $\operatorname{per}\Gamma$ obtained from Γ by forward mutating its summand corresponds to the vertex i. One can equip $\mathbf{X}$ with a left Γ'-module structure by specifying an isomorphism between dg algebra

$$\mathcal{H}om_\Gamma(\mathbf{X},\mathbf{X}) \cong \Gamma', \tag{2.13}$$ eq:dg.iso

see [15, Proposition 3.5] for details.

pp:KY **Proposition 2.12.** [15, Theorem 3.2] *There is a pair of derived equivalences (inverse to each other)*

$$\mathcal{D}(\Gamma) \underset{\psi^\flat_{S'}=?\overset{L}{\otimes}_{\Gamma'}\mathbf{X}}{\overset{\psi^\sharp_S=\mathcal{H}om_\Gamma(\mathbf{X},?)}{\rightleftarrows}} \mathcal{D}(\Gamma'), \tag{2.14}$$ eq:hst

called half spherical twists *($\mathcal{H}om$ is forward and $\overset{L}{\otimes}$ is backward). Moreover, we have*

$$\mathcal{H}_{\Gamma'} = \psi^\sharp_S\left((\mathcal{H}_\Gamma)^\sharp_S\right), \quad \mathcal{H}_\Gamma = \psi^\flat_{S'}\left(((\mathcal{H}_{\Gamma'}))^\flat_{S'}\right),$$

Similarly, we have another pair of derived functors $\psi^\sharp_{S'} = (\psi^\flat_S)^{-1}\colon \mathcal{D}(\Gamma') \to \mathcal{D}(\Gamma)$ and satisfies

$$\psi^\sharp_{S'} \circ \psi^\sharp_S = \phi_S^{-1} \in \operatorname{Aut}\mathcal{D}_{fd}(\Gamma),$$
$$\psi^\flat_S \circ \psi^\flat_{S'} = \phi_{S'} \in \operatorname{Aut}\mathcal{D}_{fd}(\Gamma').$$

We need the following lemma about composition of derived functors (for dg settings).

lem:dg **Lemma 2.13.** [12, § 7.3 (c)] *Suppose we have there dg algebras Γ,Γ',Γ'' and two bimodules ${}_{\Gamma'}\mathbf{X}_\Gamma$, ${}_{\Gamma''}\mathbf{Y}_{\Gamma'}$ that induces derived equivalences*

$$\mathcal{D}(\Gamma) \underset{?\overset{L}{\otimes}_{\Gamma'}\mathbf{X}}{\overset{\mathcal{H}om_\Gamma(\mathbf{X},?)}{\rightleftarrows}} \mathcal{D}(\Gamma') \underset{?\overset{L}{\otimes}_{\Gamma''}\mathbf{Y}}{\overset{\mathcal{H}om_{\Gamma'}(\mathbf{Y},?)}{\rightleftarrows}} \mathcal{D}(\Gamma'').$$

Let $\mathbf{Z} = \mathbf{Y} \overset{L}{\otimes}_{\Gamma'} \mathbf{X}$, which is a $\Gamma'' - \Gamma-$bimodules. Then there are natural isomorphisms

$$\begin{array}{rcll} \mathcal{H}om_{\Gamma'}(\mathbf{Y}, \mathcal{H}om_\Gamma(\mathbf{X},?)) & \cong & \mathcal{H}om_\Gamma(\mathbf{Z},?) & : \mathcal{D}(\Gamma) \xrightarrow{\sim} \mathcal{D}(\Gamma''), \\ (? \overset{L}{\otimes}_{\Gamma''} \mathbf{Y}) \overset{L}{\otimes}_{\Gamma'} \mathbf{X} & \cong & ? \overset{L}{\otimes}_{\Gamma''} \mathbf{Z} & : \mathcal{D}(\Gamma'') \xrightarrow{\sim} \mathcal{D}(\Gamma) \end{array}$$

Now, we proceed to establish a categorical square and pentagon relations. First, there are square and pentagons in any cluster exchange graph $\operatorname{CEG}(\mathfrak{Q})$ in the following sense. Let (Q,W) be a (non-degenerated) quiver with potential with vertices i and j such that there is no arrows from i to j.

- If there is no arrows from j to i, then $\mu_i\mu_j(Q,W) = \mu_j\mu_i(Q,W)$. Denote such a square (with orientation) of the corresponding Ginzburg dg algebras as the square in (2.15).
- If there is exactly one arrows from j to i. then $\mu_j\mu_i(Q,W) = \mu_i\mu_j\mu_j(Q,W)$. Denote such a pentagon (with orientation) of the corresponding Ginzburg dg algebras as the pentagon in (2.15).

$$
\begin{tikzcd}
 & \Gamma_i \arrow[dr, "\mu_j"] & \\
\Gamma \arrow[ur, "\mu_i"] \arrow[dr, "\mu_j"'] & & \Gamma_{ij} \\
 & \Gamma_j \arrow[ur, "\mu_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Gamma_i \arrow[rr, "\mu_j"] & & \Gamma_* \arrow[dr, "\mu_i'"] & \\
\Gamma \arrow[ur, "\mu_i"] \arrow[drr, "\mu_j"'] & & & & \Gamma_{ij} \\
 & & \Gamma_j \arrow[urr, "\mu_i"'] & &
\end{tikzcd}
\tag{2.15}
$$
eq:45.q

For each directed path from : Γ to Γ_{ij} above, there is an induced derived equivalence $\mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma_{ij})$ as the composition of (Keller-Yang's) forward half spherical twists in (2.14). We claim the following.

thm:pentagon

Theorem 2.14. *In either the square or the pentagon case in (2.15), the two derived equivalences from $\mathcal{D}(\Gamma) \to \mathcal{D}(\Gamma_{ij})$ induced from the two directed paths are naturally isomorphic.*

Proof. We only deal with the pentagon case while the square one is similar but simpler. The canonical silting object in $\operatorname{per}\Gamma$ is denoted by $\mathbf{X} = \Gamma$ and we have a pentagon, as in (2.12) in $\operatorname{per}\Gamma$, by mutating the silting summands that correspond to vertices i and j. By specifying an isomorphism

$$\mathcal{H}om_\Gamma(\mathbf{X}_j, \mathbf{X}_j) \cong \Gamma_j, \tag{2.16}$$
eq:dg.iso.2

as in (2.13), $\mathbf{X}_j$ induces derived equivalences

$$\mathcal{D}(\Gamma) \underset{?\overset{L}{\otimes}_{\Gamma_j}\mathbf{X}_j}{\overset{\mathcal{H}om_\Gamma(\mathbf{X}_j,?)}{\rightleftarrows}} \mathcal{D}(\Gamma_j).$$

Denote by $\mathbf{X}' \to \mathbf{X}'_j \to \mathbf{X}'_{ij}$ the image of the mutation sequence $\mathbf{X} \to \mathbf{X}_j \to \mathbf{X}_{ij}$ under $\mathcal{H}om_\Gamma(\mathbf{X}_j, ?)$. Again, by specifying an isomorphism

$$\mathcal{H}om_{\Gamma_i}(\mathbf{X}'_{ij}, \mathbf{X}'_{ij}) \cong \Gamma_{ij}, \tag{2.17}$$
eq:dg.iso.3

$\mathbf{X}'_{ij}$ induces derived equivalences

$$\mathcal{D}(\Gamma_i) \underset{?\overset{L}{\otimes}_{\Gamma_{ij}}\mathbf{X}'_{ij}}{\overset{\mathcal{H}om_{\Gamma_i}(\mathbf{X}'_{ij},?)}{\rightleftarrows}} \mathcal{D}(\Gamma_{ij}).$$

Moreover, $\mathbf{X}_{ij} = \mathbf{X}'_{ij} \overset{L}{\otimes}_{\Gamma_j} \mathbf{X}_j$ inherits the left $\Gamma_{ij}-$module structure as follows:

$$\mathcal{H}om_\Gamma(\mathbf{X}_{ij}, \mathbf{X}_{ij}) \cong \mathcal{H}om_{\Gamma_i}(\mathbf{X}'_{ij}, \mathbf{X}'_{ij}) \cong \Gamma_{ij}. \tag{2.18}$$
eq:dg.iso.4

Denote by ς this isomorphism $\mathcal{H}om_\Gamma(\mathbf{X}_{ij}, \mathbf{X}_{ij}) \cong \Gamma_{ij}$. Then ${}_{\Gamma_{ij}}(\mathbf{X}_{ij})_\Gamma$ induces derived equivalences

$$\mathcal{D}(\Gamma) \underset{?\overset{L}{\otimes}_{\Gamma_{ij}}\mathbf{X}_{ij}}{\overset{\mathcal{H}om_\Gamma(\mathbf{X}_{ij},?)}{\rightleftarrows}} \mathcal{D}(\Gamma_{ij}). \tag{2.19}$$
eq:hst2

by Lemma 2.13.

Similarly, going from Γ to Γ_{ij} via Γ_i and Γ_*, we will get another isomorphism $\varsigma'\colon \mathcal{H}om_\Gamma(\mathbf{X}_{ij}, \mathbf{X}_{ij}) \cong \Gamma_{ij}$ with another pair of induced derived equivalences as in (2.19) To show these derived equivalences are naturally isomorphic is equivalent to show that $\varsigma = \varsigma'$.

By [15, Proposition 3.5], the isomorphisms ς and ς' are defined by specifying the correspondence, from irreducible morphisms between indecomposable projective summands of Γ_{ij} to the morphisms in $\mathcal{H}om_\Gamma(\mathbf{X}_{ij}, \mathbf{X}_{ij})$, and extending to the rest. Therefore, to prove $\varsigma = \varsigma'$ we only need to study these irreducible morphisms that correspond to arrows of the associated quivers. Note that, the correspondence between the summands of Γ_{ij} and $\mathbf{X}_{ij}$ are already determined by the associated quivers.

Let $\Gamma = \mathbf{X} = \bigoplus_{t=1}^n X_t$ and

$$\mathbf{X}_i = X_i^\sharp \oplus \bigoplus_{t\neq i} X_t, \quad \mathbf{X}_* = X_i^\sharp \oplus Y_j \oplus \bigoplus_{t\neq i,j} X_t,$$
$$\mathbf{X}_j = X_j^\sharp \oplus \bigoplus_{t\neq j} X_t, \quad \mathbf{X}_{ij} = X_j^\sharp \oplus Y_i \oplus \bigoplus_{t\neq i,j} X_t.$$

By the pentagon relation in (2.12), $\mathbf{X}_*$ forward mutates to $\mathbf{X}_{ij}$ at $X_i^\sharp$. Note that here is a non-trivial isomorphism between the corresponding quivers, see [13, Figure 5] for details, and we have $Y_i = Y_j$ (denoted by Y). As $\bigoplus_{t\neq i,j} X_t$ never changes during any mutation in (2.12), they (and morphisms between them) will correspond to the same part in Γ_{ij} via either ς or ς'. Moreover, the dimension of irreducible morphisms from Y to $X_j^\sharp$ is one. So this irreducible morphism must correspond the unique morphism in Γ_{ij} between the projective summands that correspond to vertices i and j. Then we only need to worry about the irreducible morphisms between the new summands $X_j^\sharp \oplus Y_i$ and $\bigoplus_{t\neq i,j} X_t$. By the simple projective duality in Proposition 2.11 (cf. [15, Lemma 2.15]), it is equivalent to consider the Ext between the simples in the corresponding hearts.

Let fix $t \neq i, j$ and consider the full sub-quiver with vertices i, j and t. Depending on the numbers of arrows between i, j and t, there are several cases. Let us prove one for demonstration and the rest can be done in a similar way. Suppose that there are a arrows from t to i and b arrows from t to j.

The changes between simples (and Ext^1 between them, denoted by arrows) □

cor:ST **Corollary 2.15** (Seidel-Thomas). *Suppose we are in the square or pentagon case in* (2.15). *Let S_i and S_j be the simple $\Gamma-$modules in $\mathcal{D}(\Gamma)$ that correspond to vertices i and j. Then the spherical twists ϕ_{S_i} and ϕ_{S_j} (in $\operatorname{Aut}\mathcal{D}(\Gamma)$) satisfy the commutative relation in the square case and satisfy the braid relation in the pentagon case.*

Proof. □

Definition 2.16. For a mutation equivalent class $\mathfrak{Q}$ of quivers with potential, define the auto-equivalence groupoid $\mathcal{A}ut(\Gamma(\mathfrak{Q}))$ to be the groupoid whose objects are $\Gamma(Q, W)$ for any quiver with potential (Q, W) in $\mathfrak{Q}$ and whose morphisms are the derived equivalences between them.

Construction 2.17. Now, we define a representation $\xi_\Gamma\colon \mathcal{CEG}(\mathfrak{Q}) \to \mathcal{A}ut(\Gamma(\mathfrak{Q}))$ as follows:

- $\xi_\Gamma(Q,P) = \mathcal{D}_{fd}(\Gamma(Q,P))$ for each object (i.e. a quiver with potential) (Q,W) in $\mathcal{CEG}(\mathfrak{Q})$;
- For each generating morphism (i.e. a forward mutation) $\mu\colon (Q,W) \to (Q',W')$, let $\xi_b(\eta)$ be the forward half spherical twist ψ_S in (2.14).

The well-definedness follows from Theorem 2.14.

sec:unp

2.7. **Unpunctured marked surface case.** For a marked surface $\mathbf{S}$ (say unpunctured for now), the quivers with potential associated to triangulations of $\mathbf{S}$ (in the sense of Example 2.1) form a mutation equivalent class $\mathfrak{Q}_\mathbf{S}$. One can identify $\mathrm{EG}^\circ(\mathbf{S})$ with the exchange graph of triangulations (see [8]):

$$\begin{array}{ccc} \mathrm{EG}^\circ(\mathbf{S}) & \cong & \mathrm{CEG}(\mathfrak{Q}_\mathbf{S}) \\ \mathrm{T} & \mapsto & (Q_\mathrm{T}, W_\mathrm{T}). \end{array}$$

In fact, more is true in this case. There is one canonical cluster category $\mathcal{C}(\mathbf{S})$ associated to $\mathbf{S}$ (cf. [27]) so that there is bijection between reachable indecomposables in $\mathcal{C}(\mathbf{S})$ and simple (open) arcs in $\mathbf{S}$). Moreover, this bijection induces the correspondence $\mathrm{CEG}(\mathcal{C}_\mathbf{S}) \cong \mathrm{EG}^\circ(\mathbf{S})$, sending a cluster sets $\mathbf{L}$ to a triangulation T whose arcs are the ones correspond to the (reachable) indecomposables in $\mathbf{L}$. Their associated quivers with potential also match under this correspondence: $(Q_\mathrm{T}, W_\mathrm{T}) = (Q_\mathbf{L}, W_\mathbf{L})$. Therefore, $\mathrm{EG}^\circ(\mathbf{S})$ should be considered as a special case of $\mathrm{CEG}(\mathfrak{Q}_\mathbf{S})$

Theorem 2.18. [24, Theorem 7,7] *The image of the two representations of $\mathcal{CEG}(\mathbf{S})$ is isomorphic.*

sec:double

3. Clusters of curves on twisted surfaces

3.1. **The branched double cover.** In this section, we introduce the twisted surface associated to an ideal triangulation of an unpunctured marked surface and show that it is the branched double cover of the original surface branching at the centers of each triangle.

def:xms

Definition 3.1. Let T be an ideal triangulation T of $\mathbf{S}$, The *twisted surface* Σ_T (with marked points and punctures) is defined as follows.

- For each triangle T in T, construct a twisted triangle Σ_T (the blue area is the front, the green area is the back and they are glued in a twisted way).
- Denote by $\Sigma_\mathrm{T} = \bigcup_{T \in \mathrm{T}} \Sigma_T$ the *twisted surface* of $\mathbf{S}$ with respect to T, which is obtained by gluing all Σ_T along the arcs in T.

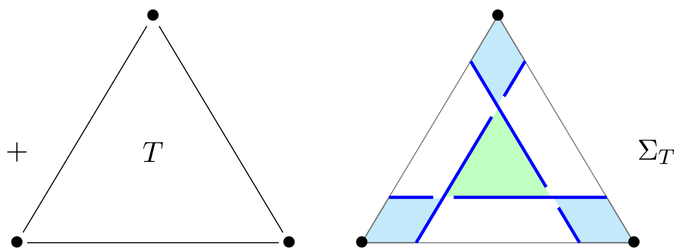


FIGURE 13.

fig:GT

For instance, Figure 14 shows the twisted surface of a triangulated pentagon, which is a torus with one boundary.

Lemma 3.2. *Let* T *and* T' *be two triangulations of* $\mathbf{S}$*. Then the twisted surfaces* Σ_{T} *and* $\Sigma_{\mathrm{T}'}$ *are homeomorphic.*

Proof. As the exchange graph of triangulations of $\mathbf{S}$ is connected, any two triangulations are related by a sequence of flips. Thus we only need to prove the lemma for the case when T and T' are differed by a flip. To construct a homeomorphism between the two triangulations, we only need to worry about the two triangles that are involved in the flip. The local twisted sub-surfaces are both annulus, as shown in Figure 20, which implies the assertion. □

There is another way to see any twisted surfaces of $\mathbf{S}$ are homeomorphic to each other, via the branched double cover.

Definition 3.3. Let $\mathbf{X}$ be a surface with marked points. Define its opened surface $\mathbf{X}^\diamond$ to be

$$\mathbf{X}^\diamond = \mathbf{X} - \bigcup_{\text{marked points } M} O(M),$$

where $O(M)$ is a small open neighborhood of M.

pp:double **Proposition 3.4.** *Let* T *be a triangulation of* $\mathbf{S}$*. Then* $\Sigma_{\mathrm{T}}^\diamond$ *is a double cover of* $\mathbf{S}^\diamond$*, branching at all the centers of triangles in* T*.*

Proof. The assertion is a local one. Since the proposition holds locally from Figure 15 and hence globally. □

As $\mathbf{X}^\diamond$ is homeomorphic to $\mathbf{X}$, one can regard Σ_{T} as a branched double cover of $\mathbf{S}$.

ex:B-HO **Example 3.5** (Birman-Hilden's double cover)**.** In the case when $\mathbf{S}$ is a polygon, the double cover in Proposition 3.4 that corresponds to the triangulation in the lower picture of Figure **??** is Birman-Hilden's double cover (cf. Figure **??**).

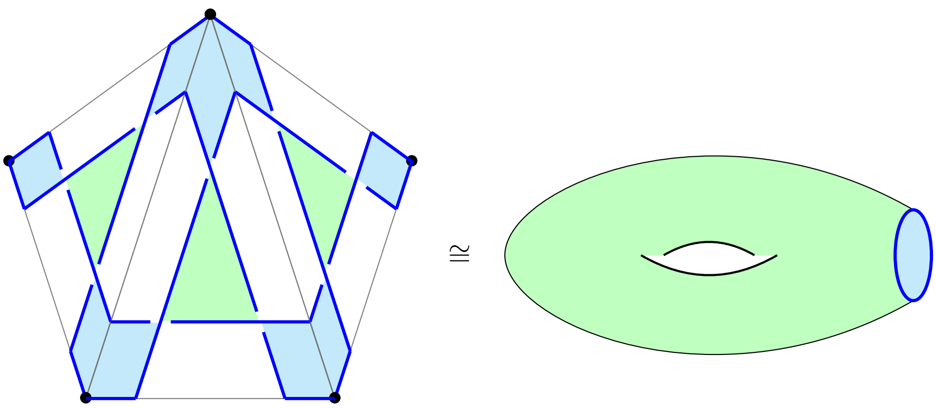


FIGURE 14. A triangulation and the twisted surface for a pentagon fig:Taitwist

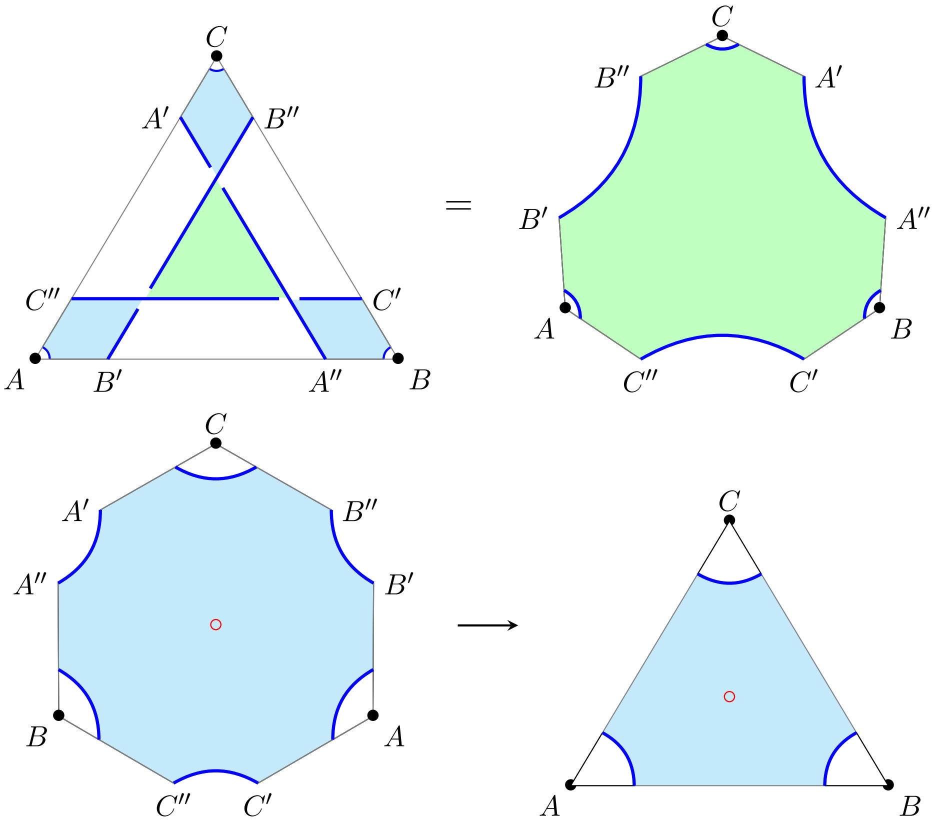


FIGURE 15. The branched double cover fig:double

Definition 3.6. The *mapping class groupoid* $\mathcal{MCG}(\Sigma_{\mathbf{S}})$ *of twisted surfaces* of $\mathbf{S}$ is the groupoid whose objects are Σ_{T} for T in $\mathrm{EG}^{\circ}(\mathbf{S}_{\triangle})$ and whose morphisms are the (isotopy classed of) homeomorphisms between them.

3.2. **Clusters of curves and tilting.**

Construction 3.7. A *(extended) cluster of curves* $\mathcal{C}$ on an oriented surfaces $\mathbf{X}$ is a collection of isotopy classes of oriented simple curves (and half-curves).

Consider an ideal triangulation T of $\mathbf{S}$ without self-folded triangles. For each triangle T in T, draw the (oriented) arcs as in Figure 16 on the twisted triangle Σ_T of Σ_{T}. Note that all the endpoints of the arcs are the midpoints of the corresponding line segments in Σ_{T}. Then,

- each arc a in T corresponds to two arcs on Σ_{T}, of which form a close curve C_a (cf. Figure 16).
- each boundary arc f of $\mathbf{S}$ corresponds to a half-curve A_f on Σ_{T}.

The *canonical clusters of curves* $\mathbf{C}_{\mathrm{T}}$ associated to such an ideal triangulation T on Σ_{T} consist of the homotopy classes of the curves $\{C_a \mid \text{arc } a \text{ in T}\}$, i.e.

$$\mathbf{C}_{\mathrm{T}} = \{[C_a] \mid \text{arc } a \text{ in T}\}. \qquad (3.1)$$

eq:cc1

Moreover, the extended cluster of curves $\widetilde{\mathbf{C}}_{\mathrm{T}}$ is defined to be

$$\widetilde{\mathbf{C}}_{\mathrm{T}} = \mathbf{C}_{\mathrm{T}} \cup \{[A_f] \mid \text{boundary arc } f \text{ in } \mathbf{S}\}. \tag{3.2}$$ eq:cc2

rem:dual1 **Remark 3.8.** When regarding Σ_{T} is the branched double cover of $\mathbf{S}$, with respect to T, the cluster of curves on Σ_{T} is the lifts of the dual T^* of the triangulation T on $\mathbf{S}$ (cf. Figure 17).

Next, we introduce an operation *tilting* on clusters of curves, which is the analogue of the (simple) tilting on hearts in triangulated categories. Note that everything works also for extended clusters of curves. However for simplicity, we only give the statement for clusters of curves.

def:Dehn1 **Definition 3.9.** For a cluster of curves $\mathbf{C}$ and an element α in $\mathbf{C}$, define the forward tilt of $\mathbf{C}$ with respect to α is the cluster of curves

$$\begin{aligned}\mathbf{C}^\sharp_\alpha \;=\; & \{[\beta] \mid \alpha \neq \beta \in \mathbf{C}, \mathrm{AI}(\beta,\alpha) \geq 0\} \,\cup\, \{[\overline{\alpha}]\} \,\cup \\ & \{[\mathrm{D}_\alpha^{-1}(\beta)] \mid \beta \in \mathbf{C}, \mathrm{AI}(\beta,\alpha) < 0\},\end{aligned}$$

where $\overline{\alpha}$ has the same homotopy class with α but different orientation. Similarly, define the backward tilt $\mathbf{C}$ with respect to α is the cluster of curves

$$\begin{aligned}\mathbf{C}^\flat_\alpha \;=\; & \{[\beta] \mid \alpha \neq \beta \in \mathbf{C}, \mathrm{AI}(\beta,\alpha) \leq 0\} \,\cup\, \{[\overline{\alpha}]\} \,\cup \\ & \{[\mathrm{D}_\alpha(\beta)] \mid \beta \in \mathbf{C}, \mathrm{AI}(\beta,\alpha) > 0\}.\end{aligned}$$

3.3. **A representation and a covering via twisted surfaces.** For any simple closed curve C on a surface $\mathbf{X}$, there is the (positive) *Dehn twist* D_C along C, which is shown in Figure **??**. This is the analogue of braid twist of an arc. We also have the analogue formula of (1.2)

$$\mathrm{D}_{\rho(\eta)} = \rho \circ \mathrm{D}_C \circ \rho^{-1},$$

for any $\Psi \in \mathrm{MCG}(\mathbf{S}_\triangle)$.

Definition 3.10. Let α and β be any two transverse (oriented) curves or arcs an oriented surface $\mathbf{X}$ and $[\alpha], [\beta]$ be the isotopy classes of them.

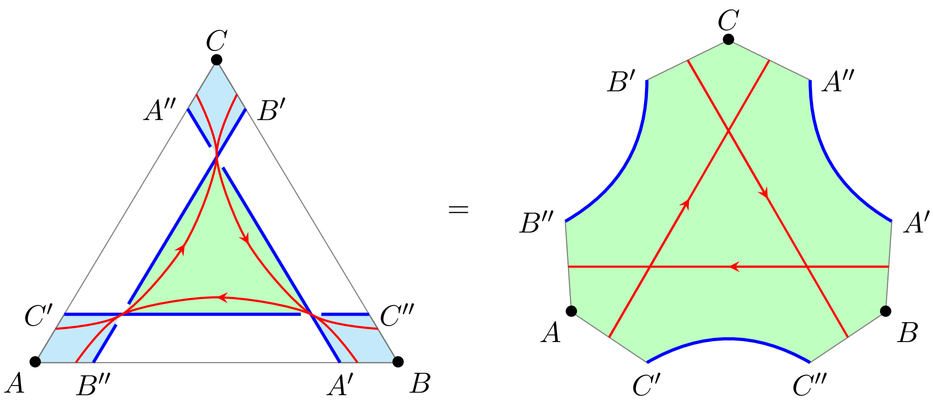


FIGURE 16. The curves on Λ_T fig:curves

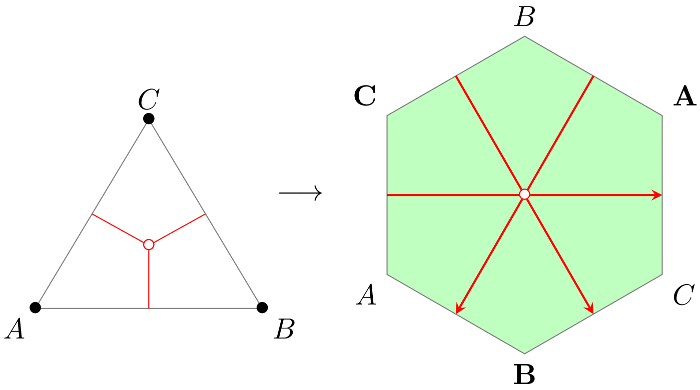


FIGURE 17. The lifting of the dual graph of the triangulation fig:lift

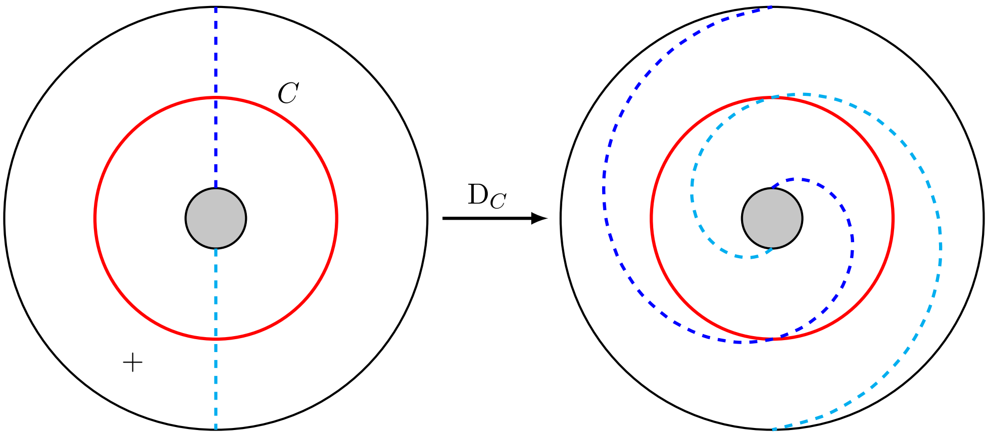


FIGURE 18. The Dehn twist fig:The Dehn tw

- An intersection of α and β is of index plus one if the orientation of the intersection agrees with the orientation of $\mathbf{X}$ (see Figure 19), and is mins one otherwise. Denote by $\mathrm{Int}^{\pm}(\alpha, \beta)$ the number of positive/negative intersections of α and β.
- The *algebraic intersection number* $\mathrm{AI}(\alpha, \beta)$ is defined to be
$$\mathrm{Int}^{+}(\alpha, \beta) - \mathrm{Int}^{-}(\alpha, \beta).$$
- The *positive geometric intersection number* $\mathrm{GI}_{+}([\alpha], [\beta])$ is defined to be
$$\min\{\mathrm{Int}^{+}(\alpha, \beta)\} \mid \alpha' \sim \alpha, \beta' \sim \beta\}.$$
- The *geometric intersection number* $\mathrm{GI}([\alpha], [\beta])$ is defined to be
$$\min\{|\alpha' \cap \beta'| \mid \alpha' \sim \alpha, \beta' \sim \beta\}.$$

  Note that we have
$$\mathrm{GI}([\alpha], [\beta]) = \mathrm{GI}^{+}([\alpha], [\beta]) + \mathrm{GI}^{+}([\beta], [\alpha]).$$

- The *algebraic intersection number* $\mathrm{AI}([\alpha],[\beta])$ is defined to be $\mathrm{AI}(\alpha,\beta)$. Note that this is well-defined and we have
$$\mathrm{GI}([\alpha],[\beta]) \geq \mathrm{GI}^{+}([\alpha],[\beta]) - \mathrm{GI}^{+}([\beta],[\alpha]) \mid \mid \mathrm{AI}([\alpha],[\beta]) \mid \mid .$$

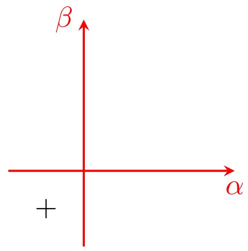


FIGURE 19. The index of an intersection fig:index

Note that AI is an anti-symmetric bilinear form on the *first homotopy class* $\mathbf{H}_1(\mathbf{X})$ of $\mathbf{X}$ and thus behaves nicely. But GI is usually much harder to calculate. Recall a result from topology.

pp:homotopy **Proposition 3.11.** [7, Proposition 1.10] *Two essential simple closed curves in a surface $\mathbf{X}$ are isotopic if and only they are homotopic.*

The following construction is the topological analogue of Proposition 2.12. Let $\mathrm{T} \xrightarrow{a} \mathrm{T}'$ be a forward flip in $\mathrm{EG}^{\circ}(\mathbf{S})$, where the arc a in T becomes the arc a' in T'. Consider the corresponding twisted surfaces $\Sigma_{\mathrm{T}}, \Sigma_{\mathrm{T}'}$ and clusters of curves $\mathbf{C}_{\mathrm{T}}, \mathbf{C}_{\mathrm{T}'}$, where the arcs a and a' correspond to the curve $C_a \in \mathbf{C}_{\mathrm{T}}$ and $C_{a'} \in \mathbf{C}_{\mathrm{T}'}$ respectively.

pp:half DT **Proposition 3.12.** *There is an unique pair of (isotopy class of) homeomorphisms (inverse of each other) in $\mathcal{MCG}(\Sigma_{\mathbf{S}})$*
$$\Sigma_{\mathrm{T}} \underset{\Psi^{\flat}_{a'}}{\overset{\Psi^{\sharp}_{a}}{\rightleftarrows}} \Sigma_{\mathrm{T}'}, \tag{3.3}$$
eq:hDt

called half Dehn twists *($\Psi^{\sharp}$ is the forward one and $\Psi^{\flat}$ the backward), satisfying the following:*

(I) for any triangle T that are both in T and T', the homeomorphisms perseveres the twisted triangle Σ_T in Σ_{T} and $\Sigma_{\mathrm{T}'}$;
(II) $\mathbf{C}_{\mathrm{T}'} = \Psi^{\sharp}_{a}\left((\mathbf{C}_{\mathrm{T}})^{\sharp}_{C_a}\right)$ and $\mathbf{C}_{\mathrm{T}} = \Psi^{\flat}_{a'}\left((\mathbf{C}_{\mathrm{T}'})^{\flat}_{C_{a'}}\right)$.

Similarly, we have another pair of homeomorphisms $\Psi^{\sharp}_{a'} = (\Psi^{\flat}_{a})^{-1} \colon \Sigma_{\mathrm{T}'} \to \Sigma_{\mathrm{T}}$, for the forward flip $\mathrm{T}' \longrightarrow \mathrm{T}$. And we have
$$\Psi^{\sharp}_{a'} \circ \Psi^{\sharp}_{a} = \mathrm{D}^{-1}_{C_a} \in \mathrm{MCG}(\Sigma_{\mathrm{T}}),$$
$$\Psi^{\flat}_{a} \circ \Psi^{\flat}_{a'} = \mathrm{D}_{C_{a'}} \in \mathrm{MCG}(\Sigma_{\mathrm{T}'}).$$

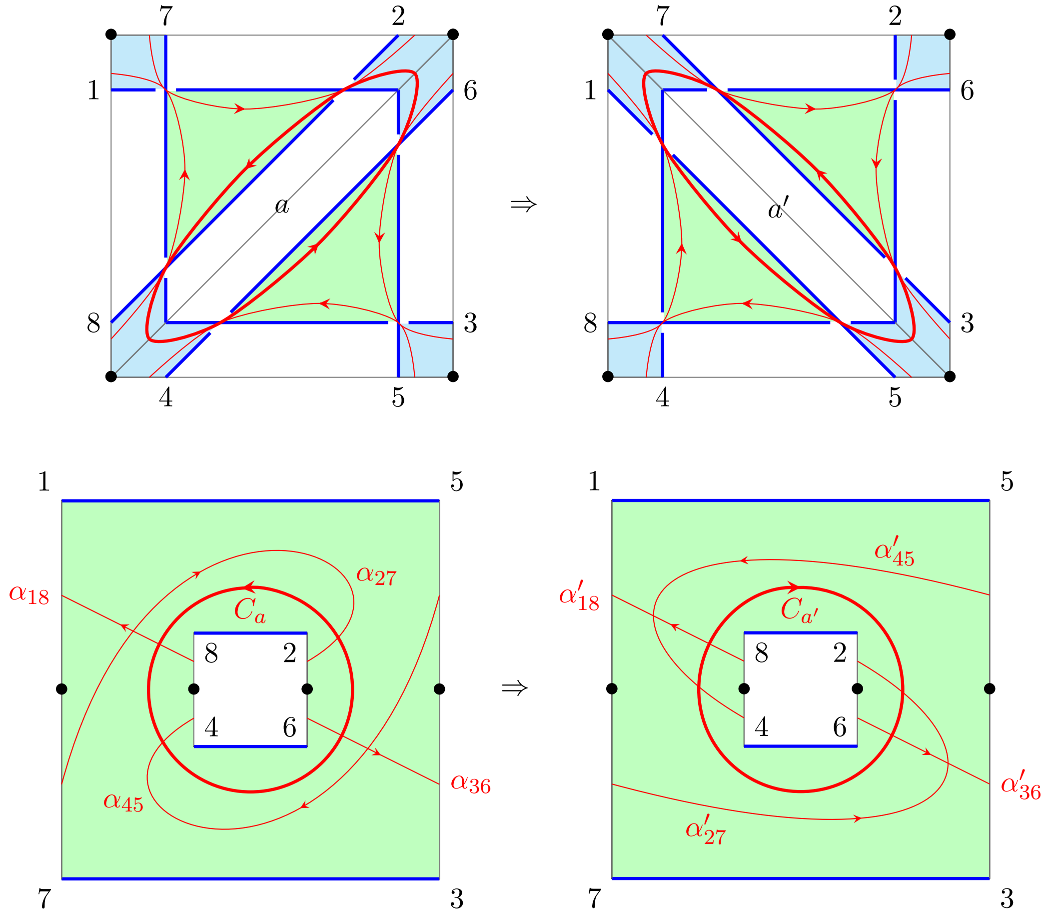


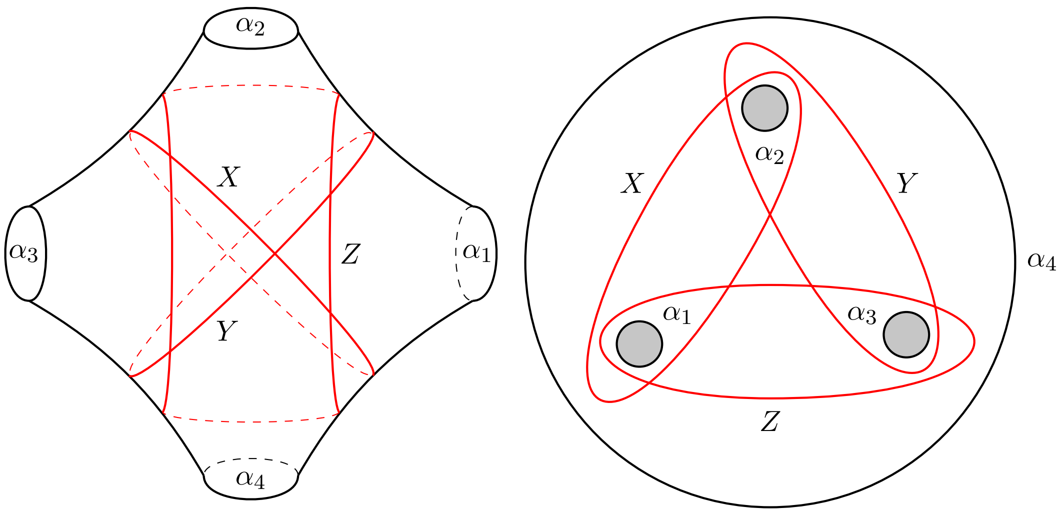
FIGURE 20. The flip and half Dehn twist fig:flip

Proof. We first show the existence of $\Psi_a^\sharp = (\Psi_{a'}^\flat)^{-1}$. By condition (I), we only need to consider the local quadrilateral where the flip happens, where the local twisted surfaces are annulus (see Figure 20). Therefore, any two homeomorphisms from Σ_T to $\Sigma_{\mathrm{T}'}$ satisfying condition (I) differ by pre-composing $\mathrm{D}_{C_a}^k$ or post-composing $\mathrm{D}_{C_{a'}}^{k'}$ for some integer k and k'. Further, requirement in condition (II) is equivalent to one of the following four equalities (by Alexander method)

$$\Psi_a^\sharp(\alpha_{18}) = \alpha'_{18}, \quad \Psi_a^\sharp(\alpha_{36}) = \alpha'_{36}\Psi_a^\sharp\left(\mathrm{D}_{C_a}^{-1}(\alpha_{27})\right) = \alpha'_{27}, \quad \Psi_a^\sharp\left(\mathrm{D}_{C_a}^{-1}(\alpha_{45})\right) = \alpha'_{45}. \quad (3.4)$$

This forces the uniqueness of $\Psi_a^\sharp$.

Similarly for the existence of $\Psi_{a'}^\sharp$. Finally, noticing that $(\mathbf{C}_\mathrm{T})^\sharp_{C_a} = \mathrm{D}_{C_a}^{-1}\left((\mathbf{C}_\mathrm{T})^\flat_{C_a}\right)$, which implies $(\Psi_{a'}^\sharp)^{-1} = \Psi_a^\flat = \Psi_a^\sharp \circ \mathrm{D}_{C_a}$. $\square$



Definition 3.13. We say a cluster of curves $\mathbf{C}$ is digon-free if $Q_A(\mathbf{C}) = Q_G(\mathbf{C})$, or equivalently, for any two (isotopy classes of) curves C, C' in $\mathbf{C}$, we have

$$\mid \mathrm{AI}(C, C') \mid = \mathrm{GI}(C, C'). \tag{3.5}$$ eq:a=g

We say $\mathbf{C}$ is *stronglydigon-free*, if any cluster of curves obtained from $\mathbf{C}$ by repeatedly tilting is also digon-free. For instance, the cluster of curves $\mathbf{C} = \{[X], [Y], [Z]\}$ that forms a lantern (cf. Figure **??**) is not digon-free. Since

$$\mathrm{AI}(C, C') = 0 \neq 2 = \mathrm{GI}(C, C'),$$

for nay C, C' in $\mathbf{C}$.

def:int.quiver **Definition 3.14.** The (geometric) intersection quiver $Q_{\mathbf{C}}$ the quiver whose vertices are in $\mathbf{C}$ and whose edges are bijective to the positive geometric intersections between the elements in $\mathbf{C}$, i.e. there are $\mathrm{GI}^+(C, C')$ arrows from C to C'.

Note that the definition of intersection quiver can be extended to a set of curves (rather than the isotopy classes). Also note that a cluster of curves is digon-free if and only if its intersection quiver is 2-cycle free.

thm:cc.0 **Theorem 3.15.** *There is a representation $\xi_d \colon \mathcal{EG}^\circ(\mathbf{S}) \to \mathcal{MCG}(\Sigma_{\mathbf{S}})$, sending an object T to its twisted surface Σ_T and a generating morphism $\mathrm{T} \xrightarrow{a} \mathrm{T}'$ to the (forward) half Dehn twist $\Psi^\sharp_{C_a}$. Moreover, there is a canonical isomorphism*

$$\begin{array}{rcl} \iota \colon Q_\mathrm{T} & \longrightarrow & Q(\mathbf{C}_\mathbf{T}) \\ \alpha & \mapsto & C_\alpha, \end{array}$$

between the FST-quiver Q_T and the intersection quiver $Q(\mathbf{C}_\mathbf{T})$ of the cluster of curves $\mathbf{C}_\mathbf{T}$ on the corresponding twisted surface Σ_T.

Proof. To prove ξ_d is well-defined, we only need to show that the half Dehn twists in $\mathcal{MCG}(\Sigma_\mathbf{S})$ satisfy the square and pentagon relations. We only deal the pentagon case while the square case is more straightforward. The issue is a local one, by condition (I)

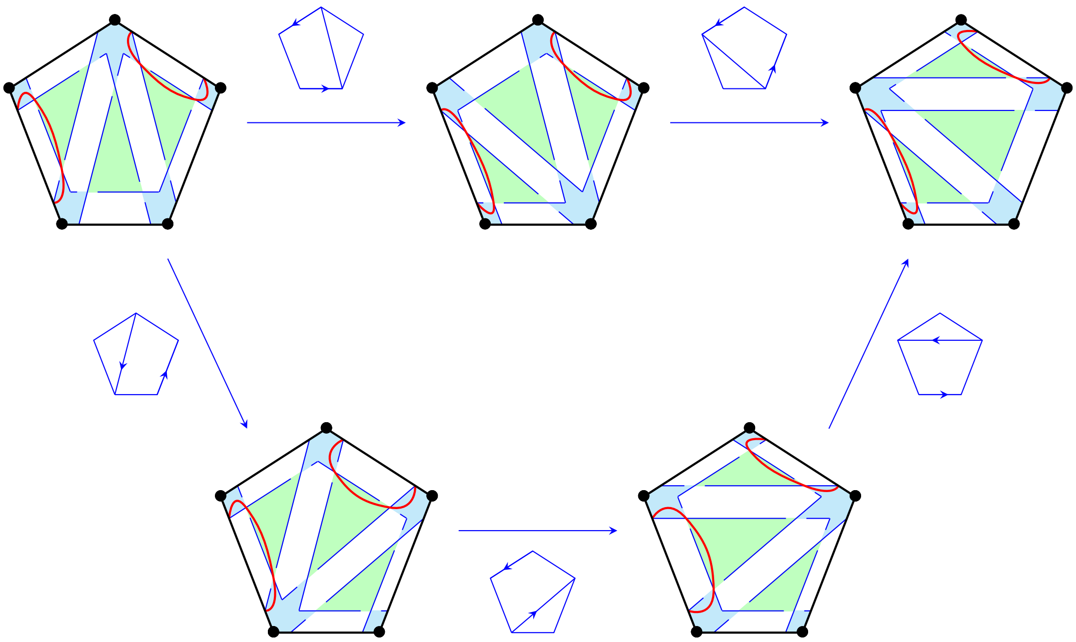


FIGURE 21. The pentagon relation

fig:5+

in Proposition 3.12. So we only need to consider the twisted surfaces correspond to a pentagon. Consider Figure 21, the twisted surfaces version of Figure 5. Note that we only need to specify the correspondence between one half curve to determine the half Dehn twist, as in the proof of Proposition 3.12. Then as we know how the red (half) curves change during half Dehn twist in Figure 21, we deduce the corresponding half Dehn twists satisfy the pentagon relation.

Moreover, the isomorphism $\iota \colon Q_{\mathrm{T}} \cong Q(\mathbf{C_T})$ is also a local statement, which can be checked for one triangle. □

A direct corollary of Theorem 3.15 is the following.

or:digon-free.0

Corollary 3.16. *The tilting of clusters of curves becomes mutation of their intersection quivers and any clusters of curves rising from triangulated marked surfaces (without punctures) are strongly digon-free.*

We can also construct a covering of $\mathcal{EG}^{\circ}(surf)$ by pulling back The Dehn twist group $\mathrm{DTG}(\mathbf{C})$ of a cluster of curves $\mathbf{C}$ on a surface $\mathbf{X}$ is the subgroup of $\mathrm{MCG}(\mathbf{X})$ generating by the Dehn twist $\{\mathrm{D}_{[C]} \mid [C] \in \mathbf{C}\}$. The point group of $\Sigma_{\mathbf{T}}$ in $\mathrm{MCG}_{\mathbf{S}}(\Sigma)$ is exactly the Dehn twist group $\mathrm{DTG}(\mathbf{C_T})$ of the corresponding cluster of curves $\mathbf{C_T}$. Then we have another conjecture following Conjecture **??**

conj:twist

Conjecture 3.17. *There is a canonical isomorphisms between the twist groups*

$$\begin{array}{rcl} \mathrm{DTG}(\mathbf{C_T}) & \simeq & \mathrm{STG}(\Gamma_{\mathrm{T}}), \\ \mathrm{D}_{[C_\alpha]} & \mapsto & \phi_{S_\alpha}, \end{array}$$

where α is any arc in T, *$[C_\alpha]$ the corresponding curve in the cluster of curves $\mathbf{C_T}$ (on the twisted surface $\Sigma_{\mathbf{T}}$) and S_α the corresponding simple in the heart $\mathcal{H}_\Gamma$ (in the 3-CY category $\mathcal{D}_{fd}(\Gamma_{\mathrm{T}})$).*

3.4. **A representation via twisted surfaces.** Let $\Sigma_{\mathbf{S}}$ be the branched double cover of $\mathbf{S}$ and $\mathcal{MCG}(\Sigma_{\mathbf{S}})$ be its mapping class groupoid.

Construction 3.18. Now, we define a representation $\xi_t\colon \mathcal{EG}^\circ(\mathbf{S}) \to \mathcal{MCG}(\Sigma_{\mathbf{S}})$ as follows:

- For any object (i.e. a triangulation) T in $\mathcal{EG}^\circ(\mathbf{S})$, let $\xi_b(\mathrm{T})$ be the (isotopy class of) twisted surface Σ_{T}.
- For each generating morphism (i.e. a forward flip) $\eta\colon \mathrm{T} \to \mathrm{T}' = \mathrm{T}^\sharp_\gamma$, let $\xi(\eta)$ be the (isotopy class of) homeomorphism from Σ_{T} to $\Sigma_{\mathrm{T}'}$, such that
$$\xi(\eta)\left(\mathbb{T}^\sharp_\gamma\right) = \mathbb{T}',$$
where $\mathbb{T}$ and $\mathbb{T}'$ are the canonical triangulations of $\mathbf{S}_\triangle(\mathrm{T})$ and $\mathbf{S}_\triangle(\mathrm{T}')$, respectively.

As triangulations of decorated surfaces satisfy the square and pentagon relations (cf. Figure 7 and Figure 21), it is straightforward to check the representation above is well-defined. Moreover, denoted by $\mathcal{BT}(\mathbf{S}_\triangle)$ the image of ξ_b. Then Lemma 1.11 implies
$$\pi_1(\mathcal{BT}(\mathbf{S}_\triangle)) \cong \mathrm{BTG}(\mathbf{S}_\triangle).$$

4. Twisted surfaces for tagged triangulations

We make the following observation.

rem:hole **Remark 4.1.** Notice that, each puncture P in $\mathbf{S}$ is inherited by $\Sigma_{\mathbf{T}}$ and hence become a hole h_P in $\mathbf{S}^\diamond$ as well as $\Sigma^\diamond_{\mathrm{T}}$. On the other hand, by construction P corresponds to a boundary component ∂_P in Σ (and in $\Sigma^\diamond_{\mathrm{T}}$), e.g. the big blue triangle in the left picture of Figure **??**. In fact, these two holes h_P and ∂_P in $\Sigma^\diamond_{\mathrm{T}}$ are the covers, in Proposition 3.4, of the hole h_P in $\mathbf{S}^\diamond$.

Denote by $\Sigma^\bullet_{\mathrm{T}}$, the surface obtained from $\Sigma^\diamond_{\mathrm{T}}$ by shrinking the holes h_P and ∂_P to punctures P_h and P_∂, respectively, for every puncture $P \in \mathbf{P}$. By Proposition 3.4, there is a branched double cover
$$\iota_{\mathrm{T}}\colon \Sigma^\bullet_{\mathrm{T}} \longrightarrow \mathbf{S}.$$

Next consider a ideal triangulation T with self-folded triangle. We need to deal with the second, third and fourth puzzle pieces (cf. Definition **??**). The naive way to extend the construction above to the second puzzle piece is shown in Figure **??**. The curve C_a' that corresponds to the arc a in this naive way is clearly bad for the obviously reason that it is not simple and we need to modify it. The correction C_a should be the 'sum' of C_a' and C_b (i.e. cf. Figure **??**). Similarly, for the third puzzle piece, the curve C_{a_i}' should be replaced by the 'sum' of C_{a_i} and C_{b_i}, where $i = 1, 2$ (cf. Figure **??**); we omit the figure for the fourth puzzle piece. Now let $\mathrm{T}T = (\mathrm{T}, \zeta)$ be a tagged triangulation on $\mathbf{S}$. Draw the curves and half-curves and arcs on $\Sigma^\diamond_{\mathrm{T}}$ for each puzzle piece with corrections as above and define the (extended) cluster of curves as in (3.1) and (3.2). It is straightforward to that these curves and half-curves are simple.

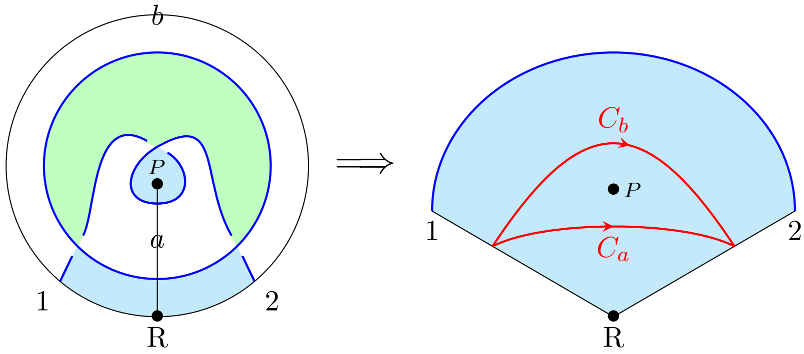


FIGURE 22. Half curves in a self-folded triangle fig:arc.self

def:cc **Definition 4.2.** Let $\mathrm{T} = (\mathrm{T}, \zeta)$ be a tagged triangulation on $\mathbf{S}$. The *cluster of curves* $\mathbf{C}_{\mathbf{T}}$ on $\Sigma_{\mathbf{T}}$ is the one inherited from the cluster of curves $\mathbf{C}_{\mathbf{T}}^{\diamond}$ on $\Sigma_{\mathrm{T}}^{\diamond}$. Similar for the extended cluster of curves $\widehat{\mathbf{C}}_{\mathrm{T}}$.

Next, we show that the definition above is well defined on $\mathrm{EG}^{\times}(\mathbf{S})$ (cf. (**??**)). This is equivalent to say, under the canonical identification (cf. Figure **??**), the cluster of curves $\mathbf{C}_{\mathrm{T}_1}$ coincides with $\mathbf{C}_{\mathrm{T}_2}$, for two equivalent tagged triangulation T_1 and T_2. This follow from Figure **??** and Figure **??**, that the effect of a filling different hole is swapping the labeling of the corresponding two curves (C_a and C_b for the second puzzle piece; C_{a_i} and C_{b_i} for the thrid puzzle piece).

Equivalently, we can draw the half-curves for C_a and C_b as in Figure 22.

rem:dual2 **Remark 4.3.** Recall that there is a canonical bijection $\mathcal{X}$ between the set of simple tagged arcs α in $\mathbf{S}$ and the set of reachable indecomposable objects X_α in the cluster category $\mathcal{C}(\Gamma_{\mathbf{S}})$. In fact, for a self-folded triangle as in Figure **??**, the object in $\mathcal{C}(\mathbf{S})$ that corresponds to the arc b should be $X_a \oplus X_{a^-}$. Further, we can lift X_a and X_{a^-} to $\widetilde{X}_a$ and $\widetilde{X}_{a^-}$ (in the corresponding silting set) in the perfect derived category $\mathrm{per}(\Gamma_{\mathbf{S}})$ with the dual objects (S_a, S_{a^-}) (simples in the corresponding heart) in the 3-CY category $\mathcal{D}_{fd}(\Gamma_{\mathbf{S}})$. we should have

$$\mathrm{Hom}(\widetilde{X}_i, S_j)) = \delta_{ij}, \quad i, j \in \{a, a^-\}.$$

Since

$$\begin{cases} \mathrm{Hom}(\widetilde{X}_a, S_a - S_{a^-}) = \mathbf{k}, \\ \mathrm{Hom}(\widetilde{X}_a \oplus \widetilde{X}_{a^-}, S_{a^-}) = \mathbf{k}, \end{cases} \qquad \begin{cases} \mathrm{Hom}(\widetilde{X}_a, S_{a^-}) = 0, \\ \mathrm{Hom}(\widetilde{X}_a \oplus \widetilde{X}_{a^-}, S_a - S_{a^-}) = 0. \end{cases}$$

, the dual of $(\widetilde{X}_a, \widetilde{X}_a \oplus \widetilde{X}_{a^-})$, which corresponds to the arcs (a, b), is $(S_a - S_{a^-}, S_{a^-})$, which should correspond to the dual of (a, b), i.e. the curves C'_a and C_b. This explains the modification above, that the curve corresponding to a should be $C'_a + C_b$, which corresponds to the simple $S_a = (S_a - S_{a^-}) + S_{a^-}$.

5. GEOMETRIC REALIZATION OF POTENTIALS

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