Euler lines, nine-point circles and integrable discretisation of surfaces via the laws of physics

by

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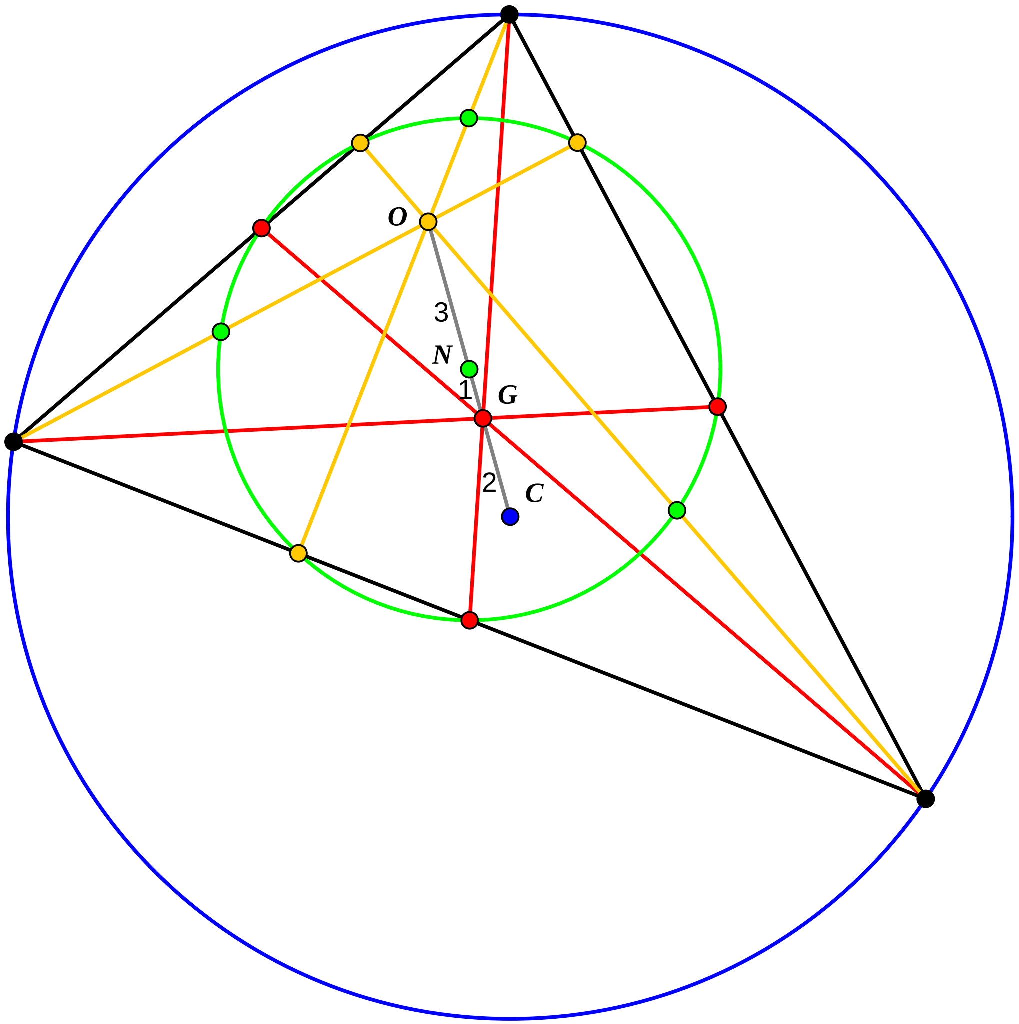
0. The Euler line and the nine-point circle



circumcentre C

centroid G

nine-point centre N

orthocentre O

Euler line:

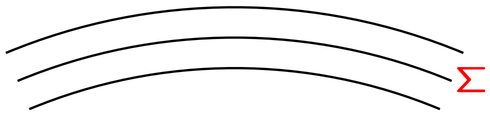
$$\overline{CG} : \overline{GN} : \overline{NO} = 2 : 1 : 3$$

Are there any canonical analogues of these objects for quadrilaterals?

1. The equilibrium equations of classical shell membrane theory

- Lamé and Clapeyron (1831): Symmetric loading of shells of revolution
- Lecornu (1880) and Beltrami (1882): Governing equations of membrane theory
- Love (1888; 1892, 1893): Theory of thin shells
- By now well-established branch of structural mechanics

Idea (see Novozhilov (1964)): Replace the three-dimensional stress tensor σ_{ik} of elasticity theory defined throughout a thin shell by statically equivalent internal forces T_{ab}, N^a and moments M_{ab} acting on its mid-surface Σ.



Vanishing of total force: $\mathsf{T}^a{}_{b;a} = h_{ab}\mathsf{N}^a, \quad \mathsf{N}^a{}_{;a} + h_{ab}\mathsf{T}^{ab} = 0$

Vanishing of total moment: $\mathsf{M}^a{}_{b;a} = \mathsf{N}_b, \quad \mathsf{T}_{[ab]} = h_{c[a}\mathsf{M}^c{}_{b]}$

Fundamental forms of Σ: $\mathrm{I} = g_{ab}dx^a dx^b, \quad \mathrm{II} = h_{ab}dx^a dx^b$

} No external forces for the time being

Definition of (shell) membranes: $\mathsf{M}_{ab} = 0$

In terms of curvature coordinates:

$$\mathrm{I} := \quad d\boldsymbol{r}^2 \quad = H^2 dx^2 + K^2 dy^2$$
$$\mathrm{II} := -d\boldsymbol{r} \cdot d\boldsymbol{N} = \kappa_1 H^2 dx^2 + \kappa_2 K^2 dy^2$$

(κ_i = principal curvatures) with the decomposition of the tangent vectors

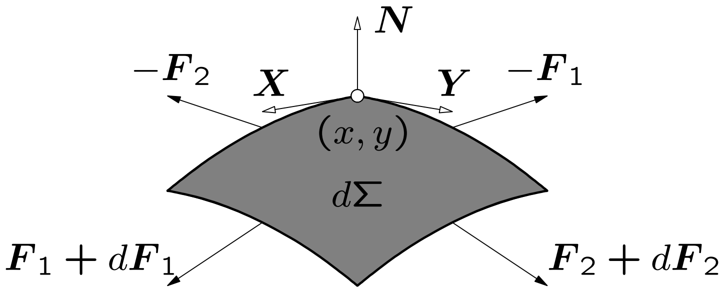
$$\boldsymbol{r}_x = H\boldsymbol{X}, \qquad \boldsymbol{r}_y = K\boldsymbol{Y}, \qquad \boldsymbol{X}^2 = \boldsymbol{Y}^2 = 1.$$

The coefficients H, K and κ_1, κ_2 obey the Gauß-Mainardi-Codazzi (GMC) equations.

Theorem: If the coefficients of two quadratic forms of the above type satisfy the GMC equations then they uniquely define a surface parametrised in terms of curvature coordinates.

3. The equilibrium conditions for membranes

$\boldsymbol{F}_1$, $\boldsymbol{F}_2$: resultant internal stresses acting on infinitesimal cross-sections $x = \text{const}$, $y = \text{const}$



Differentials:
$$d\boldsymbol{r}_1 = \boldsymbol{r}(x + dx, y) - \boldsymbol{r}(x, y)$$
$$d\boldsymbol{r}_2 = \boldsymbol{r}(x, y + dy) - \boldsymbol{r}(x, y)$$

Vanishing total force acting on $d\Sigma$: $\quad d\boldsymbol{F}_1 + d\boldsymbol{F}_2 = \boldsymbol{0}$

Vanishing total moment: $\quad d\boldsymbol{r}_1 \times \boldsymbol{F}_1 + d\boldsymbol{r}_2 \times \boldsymbol{F}_2 = \boldsymbol{0}$

Decomposition into resultant stress components per unit length according to

$$\boldsymbol{F}_1 = (T_1\boldsymbol{X} + T_{12}\boldsymbol{Y} + N_1\boldsymbol{N})K\,dy, \qquad \boldsymbol{F}_2 = (T_{21}\boldsymbol{X} + T_2\boldsymbol{Y} + N_2\boldsymbol{N})H\,dx$$

results in the membrane equilibrium equations

$$\begin{aligned} (KT_1)_x + (HS)_y + H_yS - K_xT_2 &= 0, \qquad T_{12} = T_{21} = S \\ (HT_2)_y + (KS)_x + K_xS - H_yT_1 &= 0, \qquad N_1 = N_2 = 0 \\ \kappa_1 T_1 + \kappa_2 T_2 &= 0 \end{aligned}$$

4. Vanishing 'shear stress' and constant 'normal loading'

Assumptions: • lines of principal stress = lines of curvature: $S = 0$

• additional (external) constant normal loading: $\bar{p} = \text{const}$

Equilibrium equations:

$$\begin{aligned} T_{1x} + (\ln K)_x (T_1 - T_2) &= 0 \\ T_{2y} + (\ln H)_y (T_2 - T_1) &= 0 \\ \kappa_1 T_1 + \kappa_2 T_2 + \bar{p} &= 0 \end{aligned}$$

Gauß-Mainardi-Codazzi equations:

$$\begin{aligned} \kappa_{2x} + (\ln K)_x (\kappa_2 - \kappa_1) &= 0 \\ \kappa_{1y} + (\ln H)_y (\kappa_1 - \kappa_2) &= 0 \\ \left(\frac{K_x}{H}\right)_x + \left(\frac{H_y}{K}\right)_y + HK\kappa_1\kappa_2 &= 0 \end{aligned}$$

The above system is coupled and nonlinear. Only privileged membrane geometries are possible.

Claim: The above system is integrable!

5. Classical and novel integrable reductions

- 'Homogeneous' stress distribution $T_1 = T_2 = c = \text{const}$:

$$\boxed{\mathcal{H} = \frac{\kappa_1 + \kappa_2}{2} = -\frac{\bar{p}}{2c}}$$

(Young 1805; Laplace 1806; integrable)

Constant mean curvature/minimal surfaces (modelling thin films ('soap bubbles')).

- Identification $T_1 = c\kappa_2$, $T_2 = c\kappa_1$:

$$\boxed{\mathcal{K} = \kappa_1\kappa_2 = -\frac{\bar{p}}{2c}}$$

(integrable)

Surfaces of constant Gaußian curvature governed by $\omega_{xx} \pm \omega_{yy} + \sin(\mathrm{h})\,\omega = 0$.

- Superposition $2T_1 = \lambda\kappa_2 + \mu$, $2T_2 = \lambda\kappa_1 + \mu$:

$$\boxed{\lambda\mathcal{K} + \mu\mathcal{H} + \bar{p} = 0}$$

(integrable)

Classical linear Weingarten surfaces.

6. Integrability (Rogers & WKS 2003)

Theorem: The mid-surfaces Σ of a shell membranes in equilibrium with vanishing 'shear' stress S and constant purely normal loading $\bar{p}$ constitute particular O surfaces. Accordingly, the corresponding equilibrium equations are integrable.

The large class of integrable O surfaces has been introduced only recently (WKS & Konopelchenko 2003).

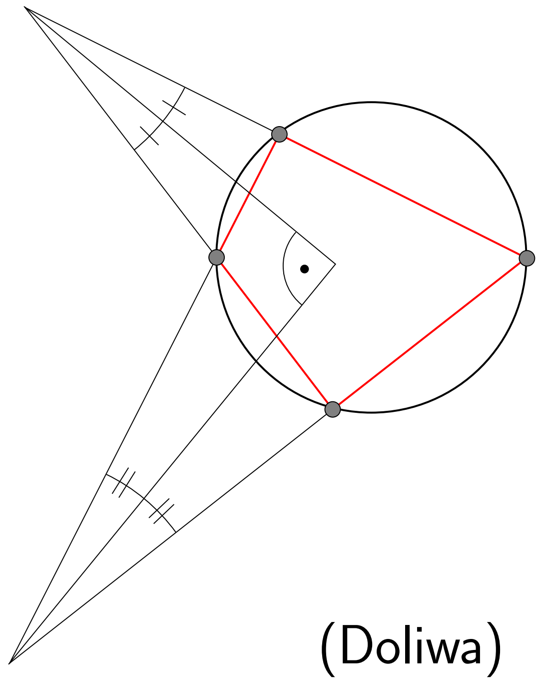
Both a Lax pair and a Bäcklund transformation for membrane O surfaces are by-products of the general theory of O surfaces.

Problem: Can shell membranes be 'discretized' in such a way that integrability is preserved?

(c.f. finite element modelling of plates and shells: 'discrete Kirchhoff techniques')

7. Discrete curvature nets ('curvature lattices')

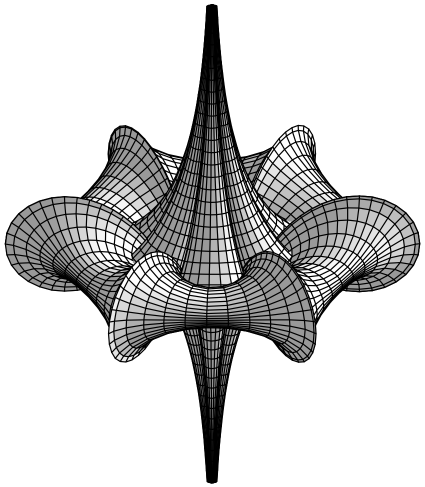
Definition: A lattice of $\mathbb{Z}^2$ combinatorics is termed a discrete curvature net if its quadrilaterals may be inscribed in circles.

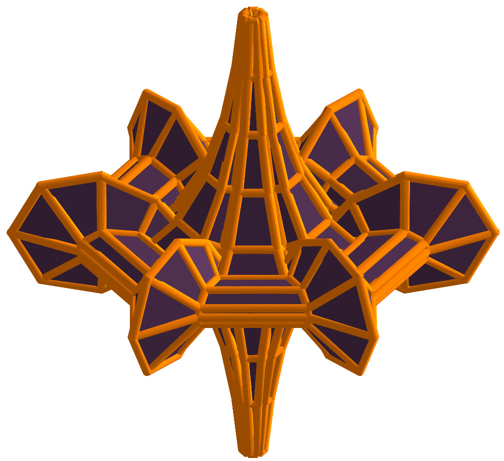


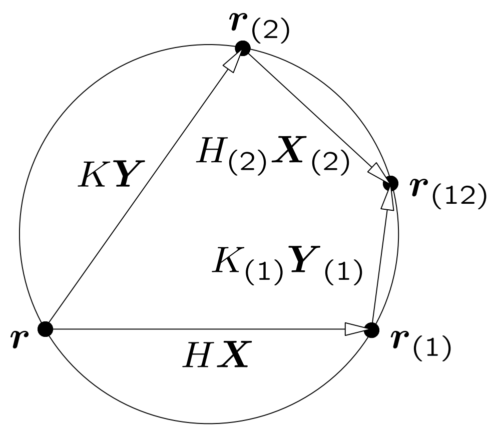
(Doliwa)

In the area of (integrable) discrete differential geometry (Bobenko & Seiler 1999) and in computer-aided surface design (Gregory 1986), the canonical discrete analogue of a 'small' patch of a surface bounded by two pairs of lines of curvature turns out to be a planar quadrilateral which is inscribed in a circle.

Application: Discrete pseudospherical surfaces (WKS 2003)







Edge vector decomposition:

$$\boldsymbol{r}_{(1)} - \boldsymbol{r} = H\boldsymbol{X}, \quad \boldsymbol{r}_{(2)} - \boldsymbol{r} = K\boldsymbol{Y}$$

Discrete Gauß equations (Konopelchenko & WKS 1998):

$$\boldsymbol{X}_{(2)} = \frac{\boldsymbol{X} + q\boldsymbol{Y}}{\Gamma}, \quad \boldsymbol{Y}_{(1)} = \frac{\boldsymbol{Y} + p\boldsymbol{X}}{\Gamma}, \quad \Gamma = \sqrt{1 - pq}$$

These imply the cyclicity condition

$$\boldsymbol{X}_{(2)} \cdot \boldsymbol{Y} + \boldsymbol{Y}_{(1)} \cdot \boldsymbol{X} = 0.$$

Closing condition:

$$H_{(2)} = \frac{H + pK}{\Gamma}, \quad K_{(1)} = \frac{K + qH}{\Gamma} \tag{1}$$

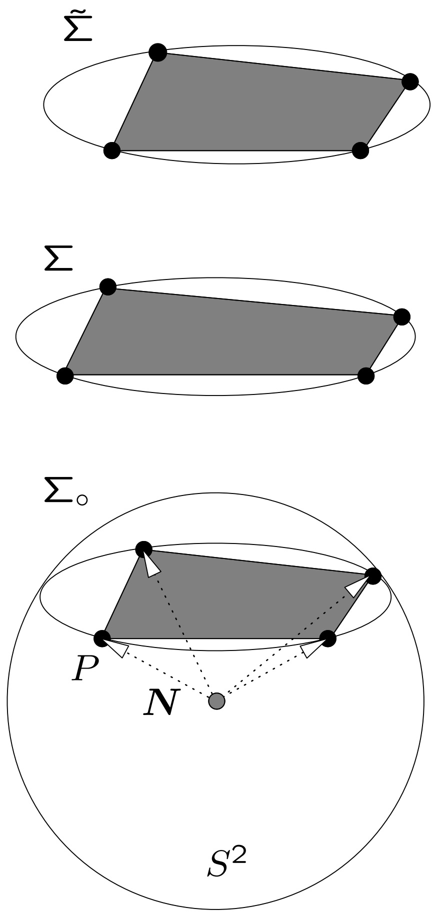
9. Discrete Combescure transforms and Gauß maps (Konopelchenko & WKS 1998)

A discrete surface $\tilde{\Sigma}$ constitutes a discrete Combescure transform of a discrete curvature net Σ if its edges are parallel to those of Σ.

Any discrete Combescure transform $\tilde{\Sigma}$ corresponds to another solution $(\tilde{H}, \tilde{K})$ of the closing condition (1).

In particular, choose a point P on the unit sphere S^2. Then, there exists a unique discrete surface $\Sigma_\circ$ with vertices on S^2 whose edges are parallel to those of Σ.

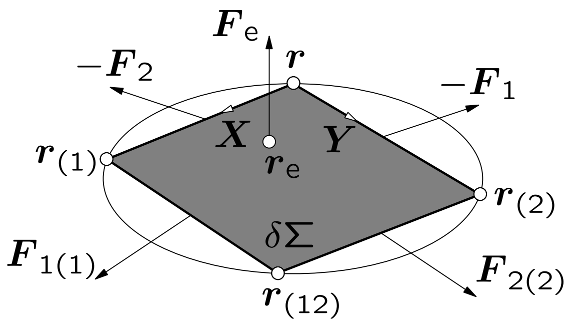
We call the discrete surface $\boldsymbol{N} : \mathbb{Z}^2 \to S^2$ a spherical representation or discrete Gauß map of Σ.



Any discrete curvature net admits a two-parameter family of spherical representations parametrized by P!

10. 'Plated' membranes (WKS 2005, 2010)

'Discrete' (plated) membrane: composed of 'plates' which may be inscribed in circles



Assumptions:
- $\boldsymbol{F}_i \perp$ edges ('$S = 0$')
- 'Constant normal loading' $\boldsymbol{F}_{\mathrm{e}} = \bar{p}\delta\Sigma\boldsymbol{N}$, $\bar{p} = \mathrm{const}$
- $\boldsymbol{F}_i$ homogeneously distributed along edges
- $\boldsymbol{F}_{\mathrm{e}}$ acts at some 'canonical' point $\boldsymbol{r}_{\mathrm{e}}$ (tbd)

Equilibrium equations:

$$\boldsymbol{F}_{1(1)} - \boldsymbol{F}_1 + \boldsymbol{F}_{2(2)} - \boldsymbol{F}_2 + \boldsymbol{F}_{\mathrm{e}} = \boldsymbol{0} \qquad \text{(force)}$$

$$\begin{aligned} &(\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(1)}) \times \boldsymbol{F}_{1(1)} - (\boldsymbol{r}_{(2)} + \boldsymbol{r}) \times \boldsymbol{F}_1 \qquad \text{(moment)} \\ + \; &(\boldsymbol{r}_{(12)} + \boldsymbol{r}_{(2)}) \times \boldsymbol{F}_{2(2)} - (\boldsymbol{r}_{(1)} + \boldsymbol{r}) \times \boldsymbol{F}_2 + 2\boldsymbol{r}_{\mathrm{e}} \times \boldsymbol{F}_{\mathrm{e}} = \boldsymbol{0} \end{aligned}$$

Claim: Plated membranes are governed by integrable difference equations!

11. The equilibrium equations

Parametrization of the forces:

$$\boldsymbol{F}_1 = \boldsymbol{Y} \times \boldsymbol{V}, \qquad \boldsymbol{V} \cdot \boldsymbol{Y} = -\frac{1}{4}\bar{p}H^2$$
$$\boldsymbol{F}_2 = \boldsymbol{U} \times \boldsymbol{X}, \qquad \boldsymbol{U} \cdot \boldsymbol{X} = -\frac{1}{4}\bar{p}K^2 \tag{2}$$

Theorem: If we make the choice

$$\boldsymbol{r}_{\mathrm{e}} = \frac{3}{2}\boldsymbol{r}_G - \frac{1}{2}\boldsymbol{r}_C \quad ??? \tag{3}$$

then the equilibrium equations for plated membranes simplify to

$$\boldsymbol{U}_{(2)} = \frac{\boldsymbol{U} + p\boldsymbol{V} - 2[(\boldsymbol{U} + p\boldsymbol{V}) \cdot \boldsymbol{Y}]\boldsymbol{Y}}{\Gamma}$$
$$\boldsymbol{V}_{(1)} = \frac{\boldsymbol{V} + q\boldsymbol{U} - 2[(\boldsymbol{V} + q\boldsymbol{U}) \cdot \boldsymbol{X}]\boldsymbol{X}}{\Gamma} \tag{4}$$

together with

$$\boldsymbol{U} \cdot \boldsymbol{Y} + \boldsymbol{V} \cdot \boldsymbol{X} = -\frac{1}{2}\bar{p}HK. \tag{5}$$

12. Geometric interpretation

Claim: Relations (2)-(5) encapsulate pure geometry!

Firstly, expansion of the quantities U and V in terms of a basis of 'normals' $\boldsymbol{N}_i$, that is

$$\boldsymbol{U} = \sum_{i=1}^{3} H_i \boldsymbol{N}_i, \quad \boldsymbol{V} = \sum_{i=1}^{3} K_i \boldsymbol{N}_i,$$

reduces the equilibrium equations (4) to

$$H_{i(2)} = \frac{H_i + pK_i}{\Gamma}, \quad K_{i(1)} = \frac{K_i + qH_i}{\Gamma}.$$

Thus, the internal forces are encoded in discrete Combescure transforms Σ_i of the discrete membrane Σ!

Note that each normal $\boldsymbol{N}_i$ corresponds to another Combescure transform $\Sigma_{\circ i}$ with 'metric' coefficients $H_{\circ i}$ and $K_{\circ i}$.

Secondly, if we combine the coefficients of the seven Combescure-related discrete surfaces Σ, Σ_i and $\Sigma_{\circ i}$ according to

$$\mathsf{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H \\ H_{\circ 1} \\ H_{\circ 2} \\ H_{\circ 3} \end{pmatrix}, \quad \mathsf{K} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \\ K \\ K_{\circ 1} \\ K_{\circ 2} \\ K_{\circ 3} \end{pmatrix}$$

then the normalisation conditions (2) and the constraint (5) become

$$\langle \mathsf{H}, \mathsf{H} \rangle = 0, \quad \langle \mathsf{K}, \mathsf{K} \rangle = 0, \quad \langle \mathsf{H}, \mathsf{K} \rangle = 0,$$

where the scalar product $\langle \quad , \quad \rangle$ is taken with respect to the matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & \mathbb{1} \\ 0 & -\bar{p} & 0 \\ \mathbb{1} & 0 & 0 \end{pmatrix}.$$

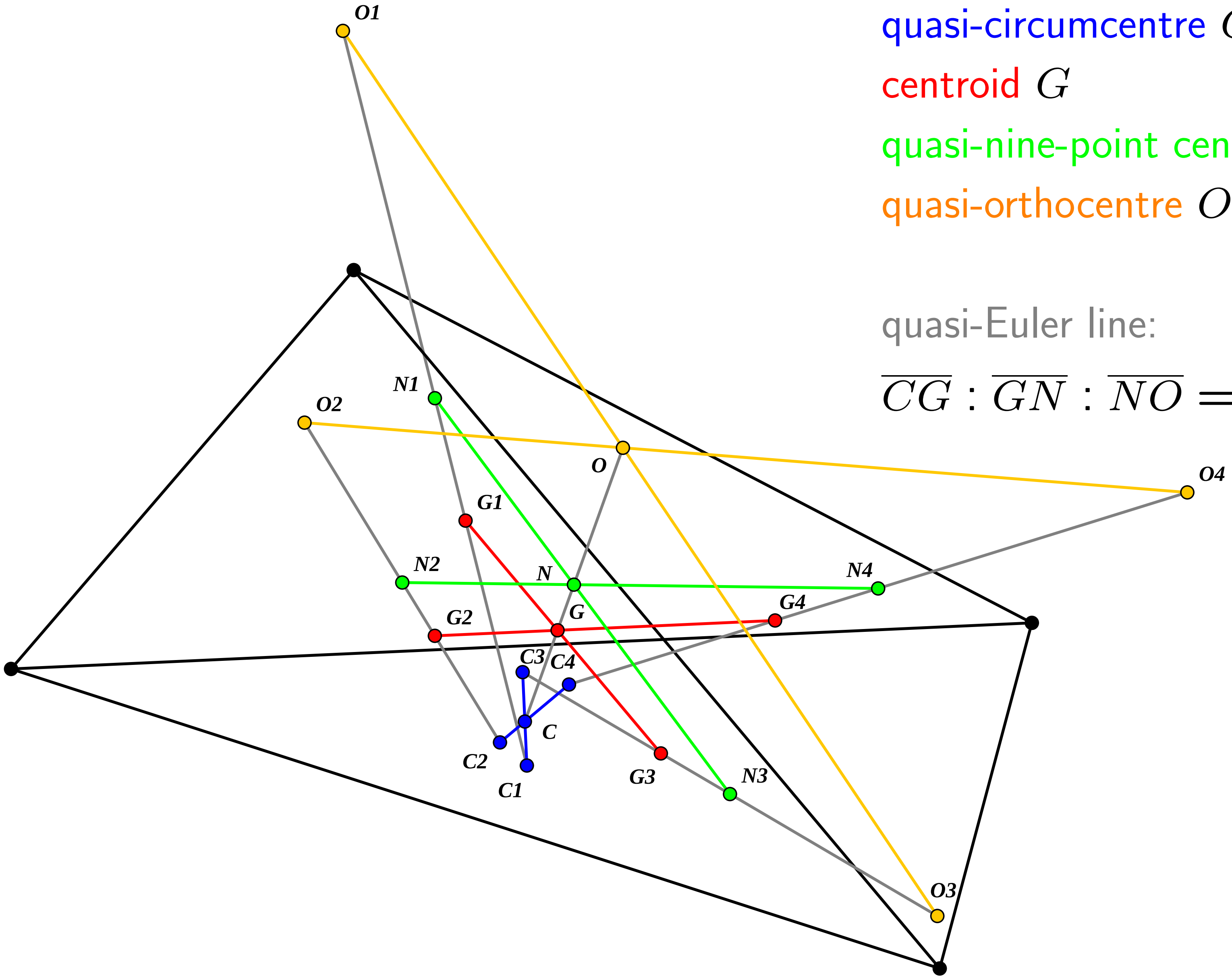
. . . .

Thus, H and K are orthogonal null vectors in a 'dual' 7-dimensional pseudo-Euclidean space with metric Λ. This observation provides the link to discrete O surface theory (WKS 2003) and implies the integrability of the equilibrium equations.

Thirdly, the 'canonical' point $\boldsymbol{r}_{\mathrm{e}}$ coincides with the quasi-nine-point centre of the corresponding cyclic quadrilateral!*

*This observation is due to N. Wildberger.

13. The quasi-Euler line (Ganin $\leq$ 2006, Rideaux 2006, Myakishev 2006)



quasi-circumcentre C

centroid G

quasi-nine-point centre N

quasi-orthocentre O

quasi-Euler line:

$$\overline{CG} : \overline{GN} : \overline{NO} = 2 : 1 : 3$$

14. A minimal surface connection

If $\bar{p} = 0$ then the discrete membrane Σ 'decouples' and constitutes an arbitrary Combescure transform of Σ.

Continuum limit for $\bar{p} = 0$:

- Only one normal and the associated Combescure transforms $\Sigma_{\circ 1}$ and Σ_1 survive.

- Equilibrium equations:

$$\langle \mathsf{H}, \mathsf{H} \rangle = \alpha(x), \quad \langle \mathsf{K}, \mathsf{K} \rangle = \beta(y), \quad \langle \mathsf{H}, \mathsf{K} \rangle = 0, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  This is the O surface representation of minimal surfaces.

- The standard discretisation of minimal surfaces (Bobenko & Pinkall 1996) admits an O surface representation with the same Λ (WKS 2003).

The 'physical' discretisation of minimal surfaces is non-standard!