Discrete Laplace-Darboux sequences, Menelaus' theorem and the pentagram map

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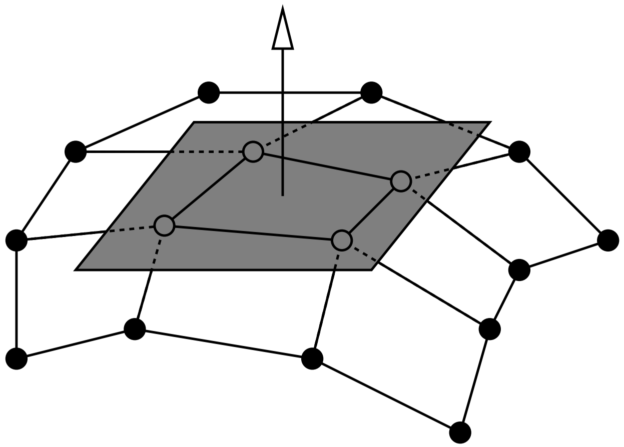
1. Discrete Laplace-Darboux transformations (Doliwa 1997)

Conjugate lattice:

$$\Phi : \mathbb{F}^2 \to \mathbb{R}^3$$

$$\mathbb{Z}^2 \cong \mathbb{F}^2 = \{(n_1, n_2) \in \mathbb{Z}^2 : n_1 + n_2 \text{ odd}\}$$

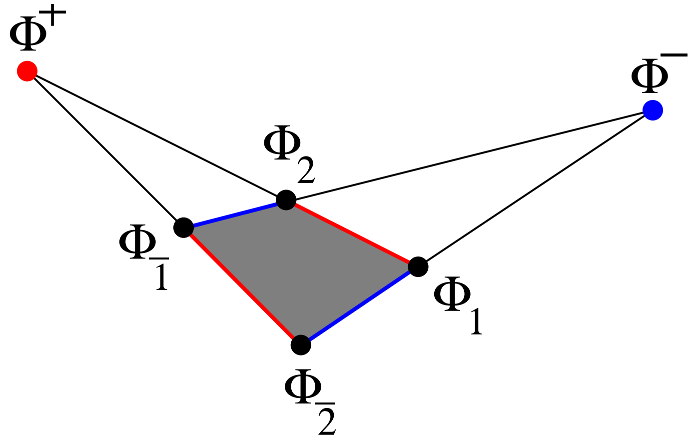
with planar faces.



Laplace-Darboux transformations:

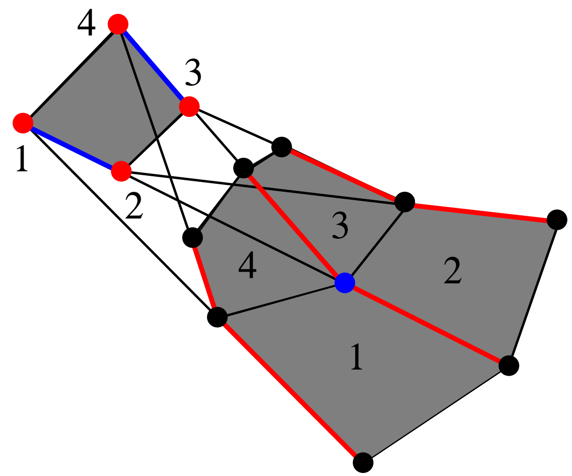
$$\mathcal{L}^+ : \; [\Phi_{\bar{2}}, \Phi_1, \Phi_2, \Phi_{\bar{1}}] \mapsto \Phi^+$$

$$\mathcal{L}^- : \; [\Phi_{\bar{2}}, \Phi_1, \Phi_2, \Phi_{\bar{1}}] \mapsto \Phi^-$$



Facts:

(1) Φ^+ and Φ^- likewise constitute conjugate lattices



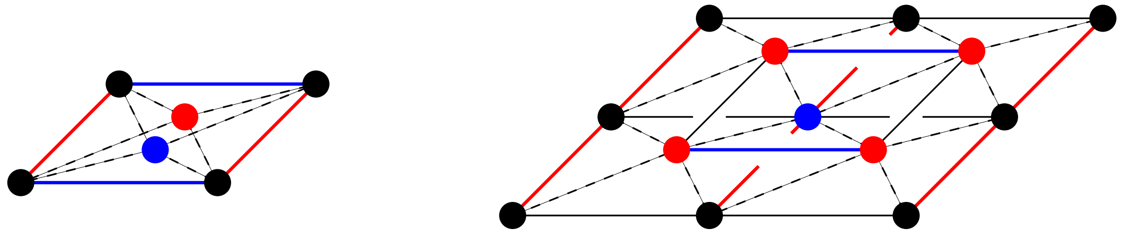
(2) "$\mathcal{L}^+ \circ \mathcal{L}^- = \mathcal{L}^- \circ \mathcal{L}^+ = \mathrm{id}$"

(3) There exist invariants $h^{(n)}$ associated with the conjugate lattices $\Phi^{(n)} = (\mathcal{L}^+)^n(\Phi)$. These obey a gauge-invariant version of the discrete 2-dimensional Toda equation, i.e. a discretisation of

$$(\ln h^{(n)})_{xy} = -h^{(n-1)} + 2h^{(n)} - h^{(n+1)}$$

3. The combinatorics of Laplace-Darboux sequences

Combinatorial picture:



Interpretation: Laplace-Darboux sequences generate three-dimensional lattices of face-centred cubic (fcc) combinatorics:

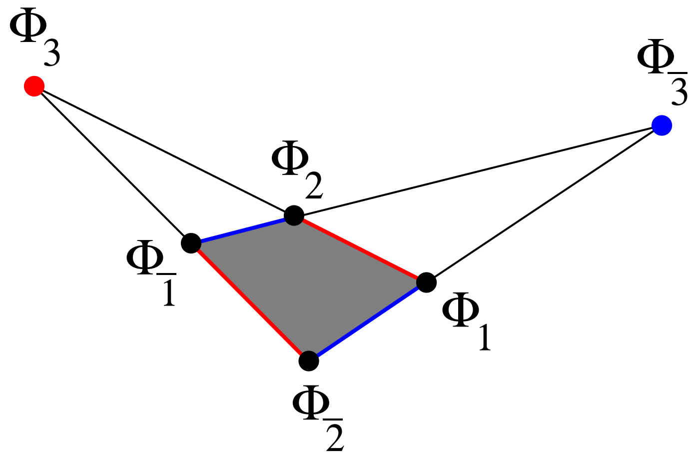
$$\Phi : \mathbb{F}^3 \to \mathbb{R}^3$$

$$\mathbb{F}^3 = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 \text{ odd}\}$$

with the properties

$$\Phi_3 = \mathcal{L}^+(\Phi_{\bar{2}}, \Phi_1, \Phi_2, \Phi_{\bar{1}})$$

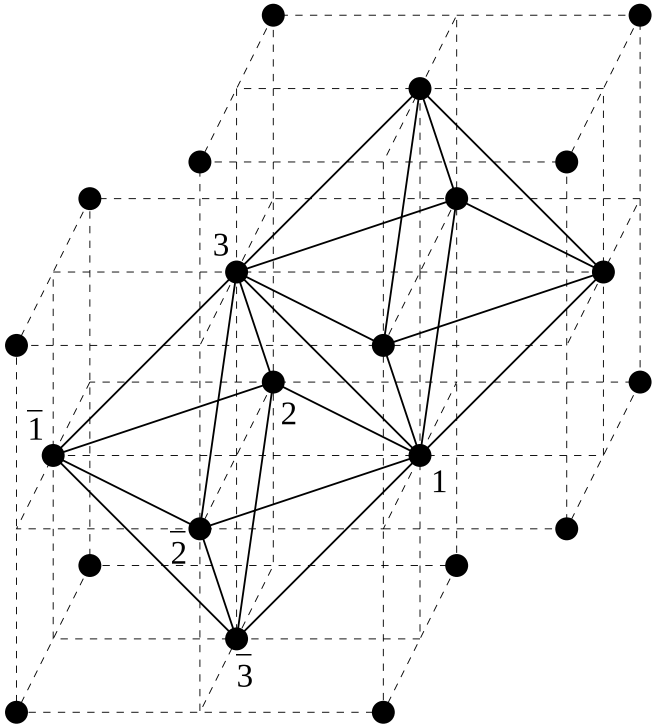
$$\Phi_{\bar{3}} = \mathcal{L}^-(\Phi_{\bar{2}}, \Phi_1, \Phi_2, \Phi_{\bar{1}})$$



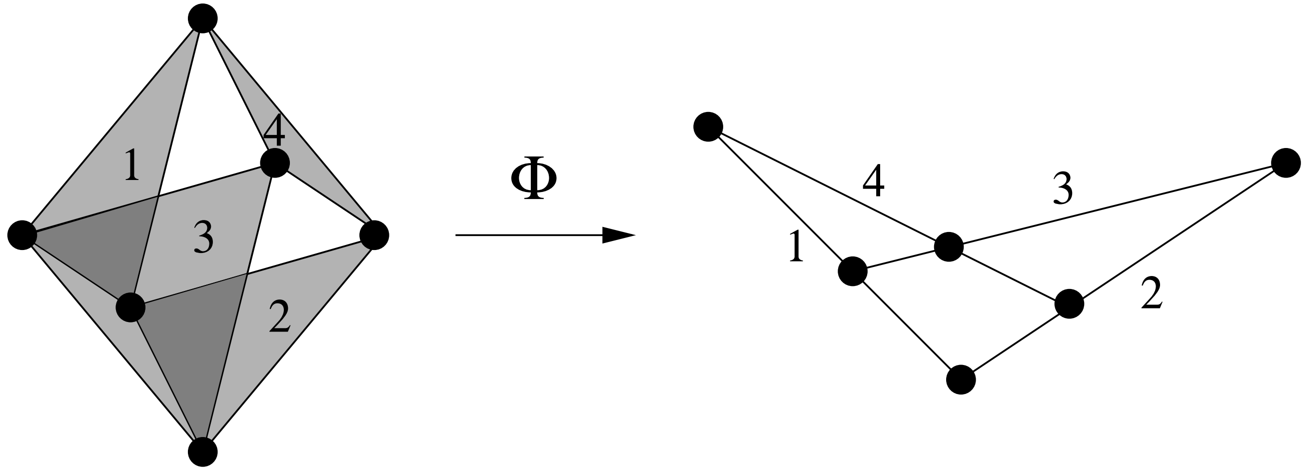
Observation: Laplace-Darboux lattices are 'symmetric' in n_1, n_2, n_3, that is the two-dimensional sublattices $\Phi(n_1 = \text{const}, n_2, n_3)$ and $\Phi(n_1, n_2 = \text{const}, n_3)$ may also be regarded as conjugate lattices which are related by Laplace-Darboux transformations!

Interpretation:

(1) $\mathbb{F}^3 =$ set of vertices of a collection of octahedra



(2) Bipartite structure of octahedra

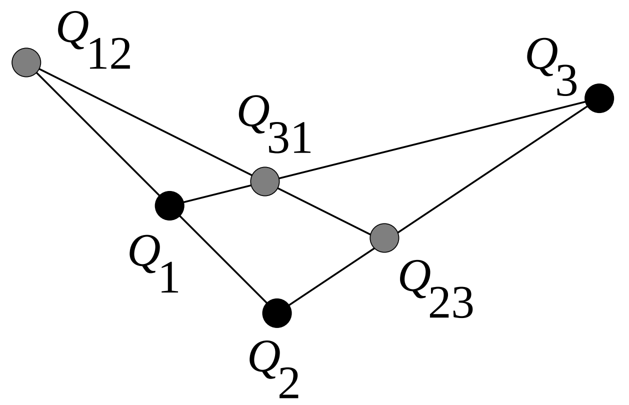


Definition. A Laplace-Darboux lattice is a map

$$\Phi : \mathbb{F}^3 \to \mathbb{R}^3 \quad (1)$$

which maps the four black faces and six vertices of any octahedron to a (planar) configuration of four lines and six points of intersection.

5. Theorem of Menelaus (100 AD; Euclid ?)



Theorem of Menelaus. Three points Q_{12}, Q_{23}, Q_{31} on the (extended) edges of a triangle with vertices Q_1, Q_2, Q_3 are collinear if and only if

$$\frac{\overline{Q_1 Q_{12}}}{\overline{Q_{12} Q_2}} \frac{\overline{Q_2 Q_{23}}}{\overline{Q_{23} Q_3}} \frac{\overline{Q_3 Q_{31}}}{\overline{Q_{31} Q_1}} = -1.$$

Conclusion: Laplace-Darboux lattices

$$\Phi : \mathbb{F}^3 \to \mathbb{R}^3$$

are characterized by the multi-ratio condition

$$\frac{\overline{\Phi_{\bar{2}} \Phi_1}}{\overline{\Phi_1 \Phi_{\bar{3}}}} \frac{\overline{\Phi_{\bar{3}} \Phi_2}}{\overline{\Phi_2 \Phi_{\bar{1}}}} \frac{\overline{\Phi_{\bar{1}} \Phi_3}}{\overline{\Phi_3 \Phi_{\bar{2}}}} = -1$$

which holds on each octahedron.

Convention: The above figure is termed Menelaus configuration.

6. The dSKP equation

Introduction of shape factors $\alpha, \beta, \gamma, \delta$ according to

$$\begin{aligned}
\Phi_{\bar{2}} - \Phi_1 &= \alpha(\Phi_1 - \Phi_{\bar{3}}) \\
\Phi_{\bar{3}} - \Phi_2 &= \beta(\Phi_2 - \Phi_{\bar{1}}) \\
\Phi_{\bar{1}} - \Phi_3 &= \gamma(\Phi_3 - \Phi_{\bar{2}}) \\
\Phi_1 - \Phi_2 &= \delta(\Phi_2 - \Phi_3) \quad \Leftrightarrow \quad \alpha\beta\gamma = -1 \text{ !!}
\end{aligned}$$

Theorem. Laplace-Darboux lattices are governed by the coupled system

$$\alpha\beta\gamma = -1, \quad \alpha_{23}\beta_{13}\gamma_{12} = -1, \quad (\alpha_{23}\gamma_{12} - 1)(\gamma + 1) = (\alpha\gamma - 1)(\gamma_{12} + 1)$$

or, equivalently, by the discrete Schwarzian KP (dSKP) equation

$$\boxed{\frac{(\phi_{\bar{2}} - \phi_1)(\phi_{\bar{3}} - \phi_2)(\phi_{\bar{1}} - \phi_3)}{(\phi_1 - \phi_{\bar{3}})(\phi_2 - \phi_{\bar{1}})(\phi_3 - \phi_{\bar{2}})} = -1}$$

for a scalar function $\phi : \mathbb{F}^3 \to \mathbb{R}$ which parametrises the shape factors according to

$$\alpha = \frac{\phi_{\bar{2}} - \phi_1}{\phi_1 - \phi_{\bar{3}}}, \quad \beta = \frac{\phi_{\bar{3}} - \phi_2}{\phi_2 - \phi_{\bar{1}}}, \quad \gamma = \frac{\phi_{\bar{1}} - \phi_3}{\phi_3 - \phi_{\bar{2}}}.$$

Alternative parametrisation:

$$\alpha = -\frac{\psi_{\bar{3}}}{\psi_{\bar{2}}}, \quad \beta = -\frac{\psi_{\bar{1}}}{\psi_{\bar{3}}}, \quad \gamma = -\frac{\psi_{\bar{2}}}{\psi_{\bar{1}}},$$

leading to the discrete modified KP (dmKP) equation

$$\boxed{\frac{\psi_{\bar{2}} - \psi_{\bar{3}}}{\psi_1} + \frac{\psi_{\bar{3}} - \psi_{\bar{1}}}{\psi_2} + \frac{\psi_{\bar{1}} - \psi_{\bar{2}}}{\psi_3} = 0.}$$

Introduction of a τ-function according to

$$\frac{\psi_{\bar{2}} - \psi_{\bar{3}}}{\psi_1} = \kappa_{[1]}\frac{\tau_{\bar{1}\bar{2}\bar{3}}\tau_1}{\tau_{\bar{2}}\tau_{\bar{3}}}, \quad \frac{\psi_{\bar{3}} - \psi_{\bar{1}}}{\psi_2} = \kappa_{[2]}\frac{\tau_{\bar{1}\bar{2}\bar{3}}\tau_2}{\tau_{\bar{1}}\tau_{\bar{3}}}, \quad \frac{\psi_{\bar{1}} - \psi_{\bar{2}}}{\psi_3} = \kappa_{[3]}\frac{\tau_{\bar{1}\bar{2}\bar{3}}\tau_3}{\tau_{\bar{1}}\tau_{\bar{2}}},$$

leading to the discrete Toda or Hirota-Miwa equation

$$\boxed{\kappa_{[1]}\tau_{\bar{1}}\tau_1 + \kappa_{[2]}\tau_{\bar{2}}\tau_2 + \kappa_{[3]}\tau_{\bar{3}}\tau_3 = 0.}$$

8. Periodic reductions

Motivation: Analogue of classical classification scheme of Laplace-Darboux sequences

Periodic reduction of the dSKP equation:

$$\phi(n_1, n_2, n_3) = \phi(n_1, n_2, n_3 + p), \quad p \text{ even}$$

Classical analogue: Periodic 2-dim Toda lattice:

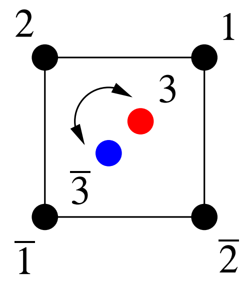
$$(\ln h^{(n)})_{xy} = -h^{(n-1)} + 2h^{(n)} - h^{(n+1)}, \quad h^{(n+p)} = h^{(n)}$$

Consistent 'quasi-periodicity' assumption:

$$\Phi(n_1, n_2, n_3) = \lambda\Phi(n_1, n_2, n_3 + p), \quad (\lambda = \text{spectral parameter!})$$

(i.e. periodicity in the setting of projective geometry.)

In the simplest case $p = 2$, we obtain for $\phi = \phi|_{n_3=0}$, $\bar\phi = \phi|_{n_3=1}$:



$\phi_{\bar 3} = \phi_3$

$$\frac{(\phi_{\bar 2} - \phi_1)(\bar\phi - \phi_2)(\phi_{\bar 1} - \bar\phi)}{(\phi_1 - \bar\phi)(\phi_2 - \phi_{\bar 1})(\bar\phi - \phi_{\bar 2})} = -1$$

$$\frac{(\bar\phi_{\bar 2} - \bar\phi_1)(\phi - \bar\phi_2)(\bar\phi_{\bar 1} - \phi)}{(\bar\phi_1 - \phi)(\bar\phi_2 - \bar\phi_{\bar 1})(\phi - \bar\phi_{\bar 2})} = -1$$

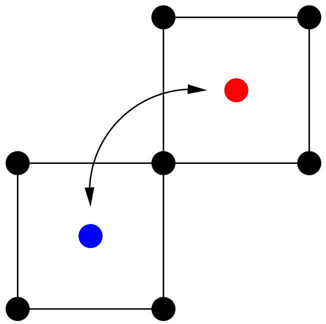
or, equivalently,

$$\frac{(\widehat\phi_{\bar 2} - \widehat\phi_1)(\widehat\phi - \widehat\phi_2)(\widehat\phi_{\bar 1} - \widehat\phi)}{(\widehat\phi_1 - \widehat\phi)(\widehat\phi_2 - \widehat\phi_{\bar 1})(\widehat\phi - \widehat\phi_{\bar 2})} = -1$$

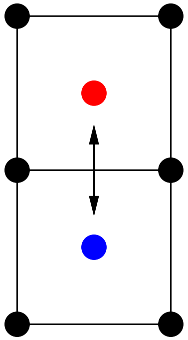
for $\{\widehat\phi\} = \{\phi\} \cup \{\bar\phi\}$.

This is a discrete Schwarzian Liouville equation (?!?) known in the theory of discrete holomorphic functions (Schramm circle patterns).

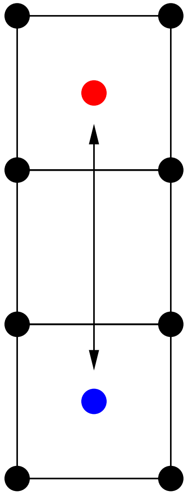
10. Period 2 + 'tangential' shifts



discrete (Schwarzian) sinh-Gordon equation (Hirota)!



discrete (Schwarzian) Korteweg-de Vries equation!



discrete (Schwarzian) Boussinesq equation!

11. The continuum limit

In general, consider the reduction

$$\tau_{\bar{3}} = T\tau_3, \quad T = T_1^{\mu} T_2^{\nu}, \quad \mu + \nu = \text{even}$$

Then, the discrete Toda equation assumes the form ($\sigma = \tau_3$)

$$\begin{aligned} \tau_{\bar{1}}\tau_1 - \tau_{\bar{2}}\tau_2 &= -\epsilon_{[1]}\epsilon_{[2]}\sigma T\sigma \\ \sigma_{\bar{1}}\sigma_1 - \sigma_{\bar{2}}\sigma_2 &= -\epsilon_{[1]}\epsilon_{[2]}\tau T^{-1}\tau. \end{aligned}$$

Continuum limit:

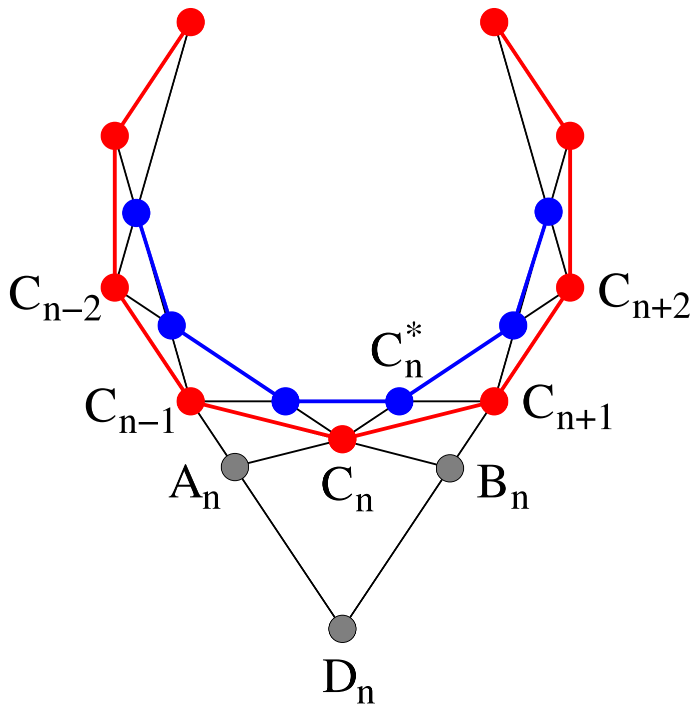
$$(\ln\tau)_{xy} = -\frac{\sigma^2}{\tau^2}, \quad (\ln\sigma)_{xy} = -\frac{\tau^2}{\sigma^2}$$

so that

$$\omega_{xy} = 4\sinh\omega, \qquad (\sigma^2/\tau^2 = \exp\omega)$$

Hence, continuum limit = sinh-Gordon equation for any T (cf. classical theory)!

Evolution of polygons on the plane (Schwartz 1992, Ovsienko, Schwartz & Tabachnikov 2009):



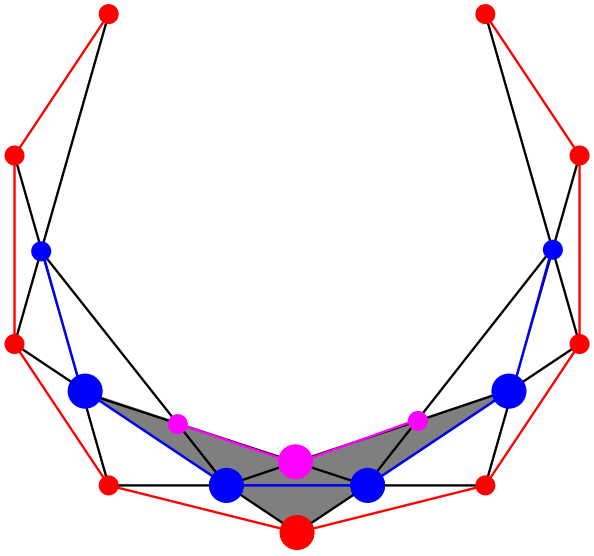
Polygon: $C : \mathbb{Z} \to \mathbb{R}^2 \quad (\mathbb{RP}^2$ in fact)

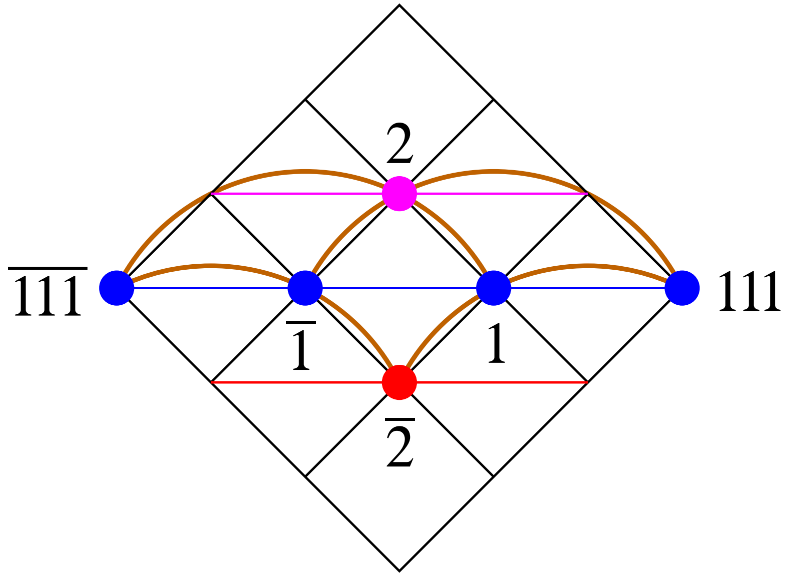
Discrete time step: $C \mapsto C^*$

Cross ratios:
$$x_n = q(D_n, A_n, C_{n-2}, C_{n-1})$$
$$y_n = q(D_n, B_n, C_{n+2}, C_{n+1})$$

Dynamical system:
$$x^*_n = x_n \frac{1 - x_{n-1}y_{n-1}}{1 - x_{n+1}y_{n+1}}$$
$$y^*_n = y_{n+1} \frac{1 - x_{n+2}y_{n+2}}{1 - x_n y_n}$$

Results: (a) Integrable if the polygon is closed (modulo a projective transformation)
(b) Boussinesq equation in the continuum limit





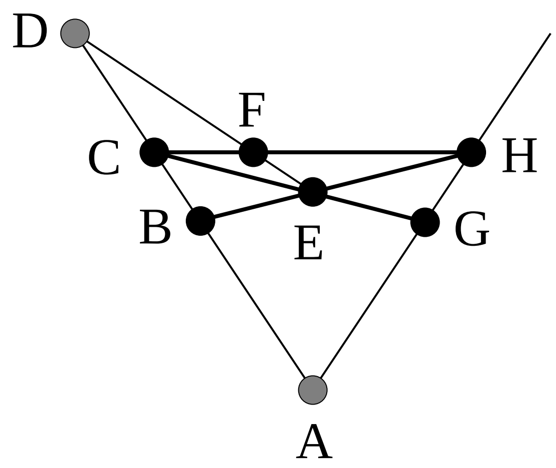
Observation: The 'pentagram lattice' is nothing but a Laplace-Darboux sequence constrained by

$$\Phi_{\bar{3}} = \Phi_{111} \quad \Leftrightarrow \quad \Phi_3 = \Phi_{\bar{1}\bar{1}\bar{1}}$$

and therefore governed by

$$\boxed{\frac{(\phi_{\bar{2}} - \phi_1)(\phi_{111} - \phi_2)(\phi_{\bar{1}} - \phi_{\bar{1}\bar{1}\bar{1}})}{(\phi_1 - \phi_{111})(\phi_2 - \phi_{\bar{1}})(\phi_{\bar{1}\bar{1}\bar{1}} - \phi_{\bar{2}})} = -1.}$$

Lemma: $q(A, B, D, C) = -M(E, G, C, F, H, B)$



Note: A is not a lattice point!

Hence:

$$x_n = -\frac{(\phi_* - \phi_*)(\phi_* - \phi_*)(\phi_* - \phi_*)}{(\phi_* - \phi_*)(\phi_* - \phi_*)(\phi_* - \phi_*)}$$

$$y_n = -\frac{(\phi_* - \phi_*)(\phi_* - \phi_*)(\phi_* - \phi_*)}{(\phi_* - \phi_*)(\phi_* - \phi_*)(\phi_* - \phi_*)}$$

and the evolution equations for x_n and y_n reduce to the above reduction of the dSKP equation!

Continuum limit: $\phi_1 = \phi + \epsilon\phi_u + O(\epsilon^2), \quad \phi_2 = \phi + \epsilon^2\phi_v + O(\epsilon^3)$

$$\boxed{\phi_{vv} - \frac{\phi_{uu}}{\phi_u^2}\phi_v^2 + \frac{3}{4}\{\phi; u\}_u \phi_u = 0}$$

Schwarzian Boussinesq equation

Note: The above discrete SBQ equation is non-standard!