Moving Boundary Problems for the Harry Dym Equation & Reciprocal Associates

Colin Rogers

Australian Research Council Centre of Excellence
for
Mathematics & Statistics of Complex Systems
&
The University of New South Wales
Sydney
Australia

Dedicated to Professor Francesco Calogero in celebration of his 80th birthday

Abstract

Moving boundary problems of generalised Stefan-Type are considered for the Harry Dym equation via a Painlevé II symmetry reduction. Exact solution of such nonlinear boundary value problems is obtained in terms of Yablonski-Vorob'ev polynomials corresponding to an infinite sequence of values of the Painlevé II parameter. The action of two kinds of reciprocal transformation on the class of moving boundary problems is described.

Background

Moving boundary problems of Stefan-Type have their origin in the analysis of the melting of solids and the freezing of liquids. The heat balance requirement on the moving boundary separating the phases characteristically leads to a nonlinear boundary condition on the temperature. The known exact solutions for standard 1+1-dimensional Stefan problems typically involve similarity reduction of the classical heat equation, Burgers' equation or their reciprocal associates, with a moving boundary $x = \gamma t^{1/2}$ wherein γ is constrained by a transcendental equation. However, the natural nonlinear analogues of Stefan problems for solitonic equations do not seem to have been previously investigated.

A Solitonic Connection

One intriguing solitonic link was made by Vasconceles and Kadanoff (1991) where, in an investigation of the Saffman-Taylor problem with surface tension, a one-parameter class of solutions was isolated in a description of the motion of an interface between a viscous and non-viscous fluid: this class was shown to be linked to travelling wave solutions of the well-known Harry Dym equation of soliton theory. Its occurrence in Hele-Shaw problems has been discussed in work of Tanveer and Fokas (1993, 1998).

In terms of the application of reciprocal transformations to moving boundary problems, an elegant integral representation developed by Calogero *et al* (1984, 2000) has recently been conjugated with reciprocal transformations to generate classes of novel exact solutions.

Cited Literature

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Moving Boundary Problems for the Harry Dym Equation

The Harry Dym equation

$$\rho_t + \rho^{-1}(\rho^{-1})_{xxx} = 0$$

arises as the base member corresponding to $n = 1$ of the solitonic hierarchy

$$\rho_t + \mathcal{E}_{n,x} = 0 \quad , \quad n = 1, 2, \ldots$$

where the flux terms $\mathcal{E}_n$ are generated iteratively by the relations

$$\mathcal{E}_n = -\int_x^\infty \rho^{-1} \; [\; \rho^{-1}\mathcal{E}_{n-1} \;]_{xxx} \; dx \quad , \quad n = 1, 2, \ldots \; ,$$

$$\mathcal{E}_0 = 1 \; .$$

[F. Calogero and A. Degasperis, *Spectral Transform and Solitons*, North Holland, Amsterdam 1982]

A Conservation Law

It is readily shown that the Dym hierarchy admits the conservation law

$$(\rho^2)_t + 2(\ \rho^{-1}\mathcal{E}_{n-1}\ )_{xxx} = 0 \quad , \quad n = 1, 2, \ldots$$

whence, in particular, the Harry Dym equation has the alternative representation

$$p_t + 2(p^{-1/2})_{xxx} = 0$$

with $p = \rho^2$ to be adopted in the sequel.

The Stefan-Type Moving Boundary Problems

Here, we consider the class of moving boundary problems

$$p_t + 2(p^{-1/2})_{xxx} = 0 \quad , \quad 0 < x < S(t) \quad , \quad t > 0$$

$$\left.\begin{array}{r} 2(p^{-1/2})_{xx} = L_m S^i \dot{S} \ , \\ p = P_m S^j \end{array}\right\} \quad \text{on} \quad x = S(t) \quad , \quad t > 0$$

$$2(p^{-1/2})_{xx}\Big|_{x=0} = H_0\, t^\delta \quad , \quad t > 0 \ ,$$

$$S(0) = 0 \ ,$$

where L_m, P_m, H_0 are assigned constants while i, j and δ are indices to be determined by admittance of a viable symmetry reduction.

Non-Standard Stefan Problems

The boundary conditions in the above are analogous with $i = j = 0$ to those of the classical Stefan problem with prescribed boundary flux on $x = 0$. Non-standard moving boundary problems of Stefan type with $i \neq 0$ arise in geo-mechanical models of sedimentation:

- J.B. Swenson *et al*, Fluvio-deltaic sedimentation: a generalised Stefan problem, *Eur. J. Appl. Math.* **11**, 433-452 (2000).

Generalised Stefan problems with variable latent heat have recently been discussed in:

- N.N. Salva and D.A. Tarzia, Explicit solution for a Stefan problem with variable latent heat and constant heat flux boundary conditions, *J. Math. Anal. Appl.* **379**, 240-244 (2011).
- Y. Zhou *et al*, Exact solution for a Stefan problem with latent heat a power function of position, *Int. J. Heat Mass. Transfer* **69**, 451-454 (2014).

Painlevé II Similarity Reduction

The Harry Dym equation

$$p_t + 2(p^{-1/2})_{xxx} = 0$$

admits a one-parameter class of similarity solutions with

$$p^{-1/2} = t^{(3n-1)/3} P(x/t^n)$$

where

$$P''' = \frac{m}{P^2} - n\xi \frac{P'}{P^3} \quad , \quad m = \frac{3n-1}{3}$$

and the prime denotes a derivative with respect to the similarity variable $\xi = x/t^n$.

Integration yields

$$PP'' - \frac{P'^2}{2} - \frac{n\xi}{P} - (m-n)\int \frac{1}{P}d\xi = \mathcal{I} \ ,$$

where $\mathcal{I}$ is an arbitrary constant. If we now set

$$w = aP_\xi \quad , \quad s = PP'' - \frac{P'^2}{2} - \frac{n\xi}{P} = \mathcal{I} + (m-n)\int \frac{1}{P}d\xi$$

together with the scaling $s = \epsilon z$ where

$$\epsilon = \pm 2a(m-n) \quad , \quad a^2 = \frac{1}{4\epsilon} \quad , \quad \epsilon > 0$$

then reduction is made to the canonical Painlevé II equation

$$\boxed{w_{zz} = 2w^3 + zw + \alpha}$$

Here, the Painlevé II parameter α is related to n by

$$\alpha = \pm\left(\frac{1-3n}{2}\right) \ .$$

Symmetry Reduction of the Moving Boundary Problems

Here, the moving boundary is taken to be $S : x = \gamma t^n$ whence, the class of nonlinear boundary value problems for the Harry Dym equation requires the solution of the Painlevé II equation

$$w_{zz} = 2w^3 + zw + \alpha \ ,$$

subject to the three constraints

$$2P_{\xi\xi}|_{\xi=1} = nL_m\gamma^{i+1} \ ,$$

$$P^{-2}\Big|_{\xi=1} = P_m\gamma^j \ ,$$

$$2P_{\xi\xi}|_{\xi=0} = H_0 \ ,$$

where $w = aP_\xi$.

The z, ξ Relation & Constraints

The independent variable z in the Painlevé II reduction is related to the similarity variable ξ via

$$\boxed{d\xi = \frac{\epsilon P dz}{m - n}}$$

It may be shown that the similarity reduction requires the relations

$$i = j = \frac{2(1-3n)}{3n} \ ,$$

$$\delta = -(n + \frac{1}{3}) \ .$$

Classical 1+1-Dimensional Stefan Problems

The known exact solutions for 1+1-dimensional moving boundary problems of Stefan-type for the classical heat equation and its Burgers or reciprocal associates are typically obtained via a symmetry reduction and with moving boundary $x = \gamma t^{1/2}$. The second order linear equation determined by this symmetry reduction admits general solution in terms of the erf function. The two arbitrary constants in this general solution together with the parameter γ in the moving boundary $x = \gamma t^{1/2}$ allow the solution of the Stefan problem subject to a transcendental constraint on γ.

Moving Boundary Problems for the Harry Dym Equation

The present class of moving boundary problems with $x = \gamma t^n$ involves symmetry reduction to Painlevé II and the latter does not admit a known exact solution involving two arbitrary constants. Thus, *prima facie*, it might be conjectured that these moving boundary value problems are not amenable to exact solution. However, remarkably two arbitrary constants arise in another manner which do indeed allow the construction of exact solutions to privileged infinite sequences of Stefan-type bvps for the Harry Dym equation. These sequences depend on the parameter n which, in turn, has been seen to be linked to the Painlevé II parameter α. In analogy with classical Stefan problems, there is a constraint on the parameter γ.

Application of the Painlevé II Bäcklund Transformation

In the sequel, consequences of the following well-known and elegant Bäcklund transformation for the Painlevé II equation (P_{II}) will be applied to the class of moving boundary problems for the Harry Dym equation:

Theorem

(Lukashevich 1971)

If $w_\alpha = w(z;\alpha)$ is a solution of P_{II} with parameter α, then

$$w_{\alpha+1} = -w_\alpha - \frac{(\alpha+\frac{1}{2})}{w_{\alpha,z} + w_\alpha^2 + \dfrac{z}{2}} \quad , \quad \alpha \neq -\frac{1}{2}$$

is a solution of P_{II} with parameter $\alpha + 1$. $\square$

In addition, if $w(z;\alpha)$ is a solution of P_{II} then $-w(z;-\alpha)$ is also a solution. This result together with iteration of the above Bäcklund transformation allows the generation of all known exact solutions of P_{II}.

Boundary Value Problems and the Nernst-Planck System

The known exact solutions of P_{II} in terms of Yablonski-Vorob'ev polynomials or Airy functions as generated by the iterated action of the Bäcklund transformation have previously been applied to solve steady state boundary value problems arising out of the Nernst-Planck system of two-ion electrodiffusion. The iterated action in this electrolytic setting is associated with quantized fluxes of the ionic species. Such discrete fluxes have been reported to arise across excitable membranes in certain biophysical contexts.

Painlevé II Reduction of the Nernst-Planck System

- C. Rogers, A. Bassom and W.K. Schief, On a Painlevé II model in steady electrolysis: application of a Bäcklund transformation, *J. Math. Anal. Appl.* **240**, 367-381 (1999).
- L.K. Bass, J. Nimmo, C. Rogers and W.K. Schief, Electrical structural of interfaces. A Painlevé II model, *Proc. Roy. Soc. London* A **466**, 2117-2136 (2010).
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- L. Bass and W.J. Moore, Electric fields in perfused nerves, *Nature* **214**, 393-394 (1967).

Rational Solutions of P_{II}: Yablonski-Vorob'ev Polynomials

The iterative action of the Bäcklund transformation on the seed solution $w = 0$ of P_{II} with $\alpha = 0$ conjugated with the invariance $w(z;\alpha) \to -w(z;-\alpha)$, generates the subsequent sequence of rational solutions

$$w_+ = \left(\ln \frac{P_{k-2}}{P_{k-1}}\right)_z \quad , \quad w_- = \left(\ln \frac{P_{k-1}}{P_{k-2}}\right)_z \quad , \quad k = 1, 2, \ldots$$

corresponding to the Painlevé II parameters $\alpha = \pm 1, \pm 2, \ldots$ where the P_k are the Yablonski-Vorob'ev polynomials determined via the quadratic recurrence relations

$$P_k P_{k-2} = z P_{k-1}^2 + 4(P_{k-1,z}^2 - P_{k-1} P_{k-1,zz}) \ ,$$

$$P_{-1} = P_{-2} = 1 \ .$$

The Yablonski-Vorob'ev Similarity Reductions

The $P(\xi)$ in the similarity reduction of the Harry Dym equation with $p^{-1/2} = t^m P(\xi)$ is connected to w by the relations

$$w = \pm\frac{1}{2}(\ln P)_z \ ,$$

Hence, corresponding to the sequences $\{w_+\},\ \{w_-\}$ of solutions of P_{II} one obtains two sequences of exact solutions for P, namely

$$P_+(z) = \mathbb{C}_{+,k}\left(\frac{P_{k-2}}{P_{k-1}}\right)^2 \quad , \quad P_-(z) = \mathbb{C}_{-,k}\left(\frac{P_{k-1}}{P_{k-2}}\right)^2$$

where $\mathbb{C}_{+,k},\ \mathbb{C}_{-,k}$ are arbitrary constants of integration.

The Boundary Conditions

Corresponding to the $P_+(z)$, the boundary conditions may be shown to yield

$$\frac{4\epsilon}{\mathbb{C}_{+,k}}\left[\left(\ln\left[\frac{P_{k-2}}{P_{k-1}}\right]\right)_{zz} \Big/ \left(\frac{P_{k-2}}{P_{k-1}}\right)^2\right]\Bigg|_{\xi=\gamma} = nL_m\gamma^{i+1} \ ,$$

$$\frac{1}{\mathbb{C}^2_{+,k}}\left(\frac{P_{k-1}}{P_{k-2}}\right)^4\Bigg|_{\xi=\gamma} = P_m\gamma^i \ ,$$

$$\frac{4\epsilon}{\mathbb{C}_{+,k}}\left[\left(\ln\left[\frac{P_{k-2}}{P_{k-1}}\right]\right)_{zz} \Big/ \left(\frac{P_{k-2}}{P_{k-1}}\right)^2\right]\Bigg|_{\xi=0} = H_0$$

with an analogous three conditions corresponding to the $P_-(z)$.

Admittance of the Boundary Conditions

In the preceding, the similarity variable ξ and the independent variable z in the P_{II} reduction are related by

$$d\xi = \frac{\epsilon P dz}{m-n}$$

whence, on integration, in turn, with $P = P_{+}(z)$ or $P = P_{-}(z)$ expressions are obtained of the type

$$\xi = \mathbb{K}_{+,k} + \epsilon \frac{\mathbb{C}_{+,k}}{(m-n)} \int \left(\frac{P_{k-2}}{P_{k-1}} \right)^2 dz \ ,$$

or

$$\xi = \mathbb{K}_{-,k} + \epsilon \frac{\mathbb{C}_{-,k}}{(m-n)} \int \left(\frac{P_{k-1}}{P_{k-2}} \right)^2 dz \ ,$$

where $\mathbb{K}_{+,k},\ \mathbb{K}_{-,k}$ are arbitrary constants.

Sequences of Solvable Moving Boundary Problems

Corresponding to a given choice of $P = P_{+z}$ or P_{-z}, the pairs of constants $\mathbb{C}_{+,k},\ \mathbb{K}_{+,k}$ or $\mathbb{C}_{-,k},\ \mathbb{K}_{-,k}$ together with an appropriate constraint on the parameter γ allow the admittance of the three boundary conditions. The sequence of boundary value problems with moving boundary $x = \gamma t^n$ which admit exact solution in terms of Yablonski-Vorob'ev polynomials via the preceding symmetry reduction correspond to

$$n = \frac{1 - 2\alpha}{3}$$

where $\alpha = \pm 1,\ \pm 2, \ldots$. Analogous sequences of exact solutions may be obtained in terms of classical Airy functions when $\alpha = \pm 1/2,\ +3/2,\ \ldots$.

A Single Application of the Bäcklund Transformation

The case with Painlevé parameter $\alpha = -1$ corresponds to a single application of the Bäcklund transformation to the trivial seed solution $w = 0$ with $\alpha = 0$, conjugated with the invariance of P_{II} under $w(z;\alpha) \to -w(z;-\alpha)$. This results in the solution of P_{II} with

$$w = \left[\ln \frac{P_0}{P_{-1}}\right]_z = \frac{1}{z}$$

corresponding to $P_0 = z, \; P_{-1} = 1$. The associated $P(\xi)$ in the similarity representation is given by

$$P = \mathbb{C}_{-,1}\left(\frac{P_0}{P_{-1}}\right)^2 = \mathbb{C}_{-,1} z^2 \ ,$$

where integration of the ξ, z relation yields

$$\xi = \mathbb{K}_{-,1} - \epsilon \, \mathbb{C}_{-,1} z^3 \ .$$

A Cubic Constraint

The boundary conditions yield

$$-\frac{4\epsilon}{\mathbb{C}_{-,1} z^4}\bigg|_{\xi=\gamma} = nL_m\gamma^{i+1} \ ,$$

$$\frac{1}{\mathbb{C}^2_{-,1} z^4}\bigg|_{\xi=\gamma} = P_m\gamma^{i} \ ,$$

$$-\frac{4\epsilon}{\mathbb{C}_{-,1} z^4}\bigg|_{\xi=0} = H_0$$

The above triad serves to determine the constants $\mathbb{C}_{-,1}$ and $\mathbb{K}_{-,1}$ while imposing a cubic constraint on $\Gamma = \gamma^{1/4}$ of the form

$$\Gamma^3 + \lambda\Gamma + \mu = 0 \ .$$

The exact solution of the class of moving boundary problems for the Harry Dym equation as generated by a single application of the Bäcklund transformation is given by the travelling wave representation

$$p = \mathbb{C}_{-,1}^{-2/3} \left[\frac{1}{\epsilon}(x - \mathbb{K}_{-,1}\ t) \right]^{-3/4} .$$

Travelling wave solutions for the classical Stefan problem associated with the melting and freezing of solids have been previously investigated in:

- J.B. Keller, Melting and freezing at constant speed, *Phys. Fluids* **92**, 2013 (1986).

Summary

The iterated action of the Bäcklund transformation for $P_{\rm II}$ may be used to generate exact solutions of increasing complexity of moving boundary problems for the Harry Dym equation and with boundaries $S(t) : x = \gamma t^n, \ n = \dfrac{1-2\alpha}{3}$ and Painlevé II parameters $\alpha = \pm 1, \ \pm 2, \ \pm 3, \ \ldots$. In the latter case, the exact solutions are expressed via the Yablonski-Vorob'ev polynomials. In a similar manner, exact solutions can be derived in terms of classical Airy functions where $\alpha = \pm 1/2, \ \pm 3/2, \ \ldots$.

Exact solutions of Stefan-type moving boundary problems for the Dym hierarchy may likewise be sought via a $P_{\rm II}$ hierarchy similarity reduction.

Application of Reciprocal Transformations

The Dym hierarchy

$$(\rho^2)_t + 2\ (\rho^{-1}\mathcal{E}_{n-1})_{xxx} = 0 \quad , \quad n = 1,\ 2,\ \ldots$$

where

$$\mathcal{E}_n = -\int_x^\infty \rho^{-1}[\ \rho^{-1}\mathcal{E}_{n-1}\ ]_{xxx}\,dx \quad , \quad n = 1,\ 2,\ \ldots$$

$$\mathcal{E}_0 = 1$$

is invariant under the reciprocal transformation

$$\left.\begin{array}{c} dx^* = \rho^2\ dx - 2(\rho^{-1}\mathcal{E}_{n-1})_{xx}\ dt \quad , \quad t^* = t \\ \rho^* = \dfrac{1}{\rho} \end{array}\right\} \qquad \mathbb{R}^*$$

- C. Rogers and M.C. Nucci, On reciprocal Bäcklund transformations and the Korteweg de Vries hierarchy, *Physica Scripta* **33**, 289-292 (1986).

The Parametric Representation

Corresponding to a similarity solution with $\rho^{-1} = t^m P(\xi)$ of the original class of Stefan problems, the reciprocally associated class of moving boundary problems has exact solution given parametrically be the relations

$$\rho^* = t^m P(\xi) \ ,$$

$$dx^* = \frac{t^{n-2m}}{P^2(\xi)} d\xi + \left[\frac{n\xi t^{n-2m-1}}{P^2(\xi)} - 2t^{m-2n} P'' \right] dt$$

$$t^* = t$$

In the reciprocal class of moving boundary problems for the classical Harry Dym equation corresponding to $n = 1$, it may be shown that the prescribed flux condition on $x = S(t)$ goes over to a Robin-type condition on the reciprocal boundary $x^* = S^*(t^*)$.

Reciprocal Link to the KdV Singularity Manifold Equation

Under the alternative reciprocal transformation

$$\left.\begin{array}{c} d\bar{x} = \rho dx - \mathcal{E}_n dt \quad , \quad \bar{t} = t \\ \bar{\rho} = \dfrac{1}{\rho} \, , \end{array}\right\} \quad \bar{\mathbb{R}}$$

for the avatar

$$\rho_t + \mathcal{E}_{n,x} = 0 \quad , \quad n = 1, 2, ...$$

of the Dym hierarchy, the latter becomes

$$\bar{\rho}_{\bar{t}} + \bar{\mathcal{E}}_{n,\bar{x}} = 0 \quad , \quad n = 1, 2, ...$$

where the $\bar{\mathcal{E}}_n$ are given iteratively by the relations

$$\bar{\rho}^{-1} \frac{\partial}{\partial \bar{x}} \left[\, \bar{\rho}^{-1} \bar{\mathcal{E}}_n \, \right] = \frac{\partial}{\partial \bar{x}} \left[\bar{\rho}^{-1} \frac{\partial}{\partial \bar{x}} \left[\bar{\rho}^{-1} \frac{\partial}{\partial \bar{x}} \; \bar{\mathcal{E}}_{n-1} \right] \right] \quad , \quad n = 1, 2...$$

$$\bar{\mathcal{E}}_0 = -\bar{\rho} \, .$$

The Reciprocal Harry Dym Equation

In the case of the Harry Dym equation correspondng to $n = 1$ its reciprocal is

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} - \frac{\partial}{\partial \bar{x}} \left[\, \bar{\rho}_{\bar{x}\bar{x}} - \frac{3}{2} \frac{\bar{\rho}_{\bar{x}}^2}{\bar{\rho}} \, \right] = 0$$

whence, with $\bar{\rho} = \phi_{\bar{x}}$, one obtains the KdV singularity manifold equation

$$\phi_{\bar{t}} / \phi_{\bar{x}} - \{\phi; \bar{x}\} = 0$$

wherein

$$\{\phi; \bar{x}\} := \left(\frac{\phi_{\bar{x}\bar{x}}}{\phi_{\bar{x}}} \right)_{\bar{x}} - \frac{1}{2} \left(\frac{\phi_{\bar{x}\bar{x}}}{\phi_{\bar{x}}} \right)^2$$

is the Schwarzian derivative.

Exact solutions of the reciprocal class of moving boundary problems in $\bar{\rho}$ may generated via the P_{II} reduction. On use of the reciprocal relations

$$d\bar{x} = \rho dx - \left[(\rho^{-1})_{xx}\rho^{-1} - \frac{1}{2}\,(\rho^{-1})_x^2\right] dt \quad , \quad \bar{t} = t \; ,$$

$$\bar{\rho} = \frac{1}{\rho}$$

these are given parametrically via

$$\bar{\rho} = \bar{t}^m P(\xi) \; ,$$

$$d\bar{x} = \bar{t}^{-m+n} P^{-1} d\xi + \bar{t}^{-m+n-1}\left[n\xi \; P^{-1} - \left(P'' P - \frac{P'^2}{2}\right)\right] d\bar{t} \; ,$$

$$\bar{t} = t \; .$$