A Numerical Criterion for Lower bounds on K-energy maps of Algebraic manifolds

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Outline

- **Formulation of the problem**: To bound the Mabuchi energy from below on the space of Kähler metrics in a given Kähler class $[\omega]$. **Tian's program '88 -'97**: In algebraic case should restrict K-energy to "Bergman metrics".

- **Representation theory** : Toric Morphisms and ***Equivariant embeddings*** .

- **Discriminants and resultants of projective varieties**: ***Hyperdiscriminants*** and Cayley-Chow forms.

- **Output**: A *complete description* of the extremal properties of the Mabuchi energy restricted to the space of Bergman metrics .

Formulating the problem

Set up and notation:

- (X^n, ω) closed Kähler manifold
- $\mathcal{H}_\omega := \{\varphi \in C^\infty(X) \mid \omega_\varphi > 0\}$

  (the space of Kähler metrics in the class $[\omega]$)

  $\omega_\varphi := \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$
- $Scal(\omega)$: = scalar curvature of ω
- $\mu = \frac{1}{V}\int_X Scal(\omega)\omega^n$

  (average of the scalar curvature)

  V=volume

Definition. (Mabuchi 1986)
The ***K-energy map*** ν_ω : $\mathcal{H}_\omega \longrightarrow \mathbb{R}$ is given by

$$\nu_\omega(\varphi) := -\frac{1}{V}\int_0^1 \int_X \dot{\varphi}_t(\mathrm{Scal}(\omega_{\varphi_t}) - \mu)\omega_t^n dt$$

φ_t is a C^1 path in $\mathcal{H}_\omega$ satisfying $\varphi_0 = 0$, $\varphi_1 = \varphi$

Observe : φ is a critical point for ν_ω iff $\mathrm{Scal}(\omega_\varphi) \equiv \mu$ (a constant)

Basic Theorem (Bando-Mabuchi, Donaldson, ..., Chen-Tian)
If *there is a* $\psi \in \mathcal{H}_\omega$ *with constant scalar curvature then there exits* $C \geq 0$ *such that*

$$\nu_\omega(\varphi) \geq -C \textit{ for all } \varphi \in \mathcal{H}_\omega \ .$$

Question $(*)$:
Given $[\omega]$ *how to detect when* ν_ω *is bounded below on* $\mathcal{H}_\omega$?

N.B. : In general we *do not know (!)* if there is a constant scalar curvature metric in the class $[\omega]$.

Special Case: Assume that $[\omega]$ is an *integral* class, i.e. there is an ample divisor $\mathbb{L}$ on X such that

$$[\omega] = c_1(\mathbb{L})$$

We may assume that $X \longrightarrow \mathbb{P}^N$ (embedded) and $\omega = \omega_{FS}|_X$

Observe that for $\sigma \in G := SL(N+1, \mathbb{C})$ there is a $\varphi_\sigma \in C^\infty(\mathbb{P}^N)$ such that

$$\sigma^*\omega_{FS} = \omega_{FS} + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi_\sigma > 0$$

This gives a map

$$G \ni \sigma \longrightarrow \varphi_\sigma \in \mathcal{H}_\omega$$

The space of ***Bergman Metrics*** is the image of this map

$$\mathcal{B} := \{\varphi_\sigma \mid \sigma \in G\} \subset \mathcal{H}_\omega \ .$$

Tian's idea: ***RESTRICT THE K-ENERGY TO*** $\mathcal{B}$

Question $(**)$:

Given $X \longrightarrow \mathbb{P}^N$ *how to detect*

when ν_ω *is bounded below on* $\mathcal{B}$*?*

Definition. Let $\Delta(G)$ be the space of *algebraic one parameter subgroups* λ of G. These are algebraic homomorphisms

$$\lambda : \mathbb{C}^* \longrightarrow G \qquad \lambda_{ij} \in \mathbb{C}[\, t \,,\; t^{-1} \,] \,.$$

Definition. (The space of ***degenerations*** in $\mathcal{B}$)

$$\Delta(\mathcal{B}) := \{\mathbb{C}^* \xrightarrow{\varphi_\lambda} \mathcal{B}\,;\; \lambda \in \Delta(G)\} \,.$$

Theorem . (Paul 2012)

Assume that for every degeneration λ in $\mathcal{B}$ there is a (finite) constant $C(\lambda)$ such that

$$\lim_{\alpha \longrightarrow 0} \nu_\omega(\varphi_{\lambda(\alpha)}) \geq C(\lambda) \ .$$

Then there is a ***uniform*** constant C such that for all $\varphi_\sigma \in \mathcal{B}$ we have the lower bound

$$\nu_\omega(\varphi_\sigma) \geq C \ .$$

Equivariant Embeddings of Algebraic Homogeneous Spaces

- G reductive complex linear algebraic group:
  $G = GL(N+1,\mathbb{C}), SL(N+1,\mathbb{C})$, $(\mathbb{C}^*)^N$,

  $SO(N,\mathbb{C})$, $Sp_{2n}(\mathbb{C})$.

- $H :=$ *Zariski* closed subgroup.

- $\mathcal{O} := G/H$ associated homogeneous space.

Definition . An ***embedding*** of $\mathcal{O}$ is an irreducible G variety X together with a G-equivariant embedding $i : \mathcal{O} \longrightarrow X$ such that $i(\mathcal{O})$ is an open dense orbit of X.

Let (X_1, i_1) and (X_2, i_2) be *two embeddings* of $\mathcal{O}$.

Definition. A ***morphism*** φ from (X_1, i_1) to (X_2, i_2) is a G equivariant regular map $\varphi : X_1 \longrightarrow X_2$ such that the diagram

$$\begin{tikzcd} & X_1 \arrow[dd, "\varphi"] \\ \mathcal{O} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & X_2 \end{tikzcd}$$

commutes.

One says that (X_1, i_1) ***dominates*** (X_2, i_2) .

Assume these embeddings are both projective (hence complete) with very ample linearizations

$$\mathbb{L}_1 \in \mathrm{Pic}(X_1)^G\ ,\ \mathbb{L}_2 \in \mathrm{Pic}(X_2)^G$$

satisfying

$$\varphi^*(\mathbb{L}_2) \cong \mathbb{L}_1\ .$$

Get *injective* map of G modules

$$\varphi^* : H^0(X_2, \mathbb{L}_2) \longrightarrow H^0(X_1, \mathbb{L}_1)$$

The adjoint

$$(\varphi^*)^t : H^0(X_1, \mathbb{L}_1)^\vee \longrightarrow H^0(X_2, \mathbb{L}_2)^\vee$$

is *surjective* and gives a rational map :

$$\begin{tikzcd}
 & X_1 \arrow[r, hook] \arrow[dd, "\varphi"] & \mathbb{P}(H^0(X_1, \mathbb{L}_1)^\vee) \arrow[dd, dashed, "(\varphi^*)^t"] \\
\mathcal{O} \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \\
 & X_2 \arrow[r, hook] & \mathbb{P}(H^0(X_2, \mathbb{L}_2)^\vee)
\end{tikzcd}$$

We abstract this situation :

1. $\mathbb{V}, \mathbb{W}$ finite dimensional rational G-modules

2. v, w *nonzero* vectors in $\mathbb{V}, \mathbb{W}$ respectively

3. Linear span of $G \cdot v$ coincides with $\mathbb{V}$ (same for w)

4. $[v]$ corresponding line through v $=$ point in $\mathbb{P}(\mathbb{V})$

5. $\mathcal{O}_v := G \cdot [v] \subset \mathbb{P}(\mathbb{V})$ (projective orbit)

6. $\overline{\mathcal{O}}_v =$ Zariski closure in $\mathbb{P}(\mathbb{V})$.

Definition . $(\mathbb{V}; v)$ *dominates* $(\mathbb{W}; w)$ if and only if there exists $\pi \in Hom(\mathbb{V}, \mathbb{W})^G$ such that $\pi(v) = w$ and the rational map $\pi : \mathbb{P}(\mathbb{V}) \dashrightarrow \mathbb{P}(\mathbb{W})$ induces a regular finite ***morphism*** $\pi : \overline{G \cdot [v]} \longrightarrow \overline{G \cdot [w]}$

$$\begin{tikzcd}
& \overline{\mathcal{O}}_v \arrow[r, hook] \arrow[dd, "\pi"] & \mathbb{P}(\mathbb{V}) \arrow[dd, dashed, "\pi"] \\
\mathcal{O} \arrow[ur, "i_v"] \arrow[dr, "i_w"'] & & \\
& \overline{\mathcal{O}}_w \arrow[r, hook] & \mathbb{P}(\mathbb{W})
\end{tikzcd}$$

Observe that the map π extends to the boundary if and only if

$$(*) \qquad \overline{G \cdot [v]} \cap \mathbb{P}(\mathrm{ker}\,\pi) = \emptyset\ .$$

- $\pi(\mathbb{V}) = \mathbb{W}$

- $\mathbb{V} = \mathrm{ker}(\pi) \oplus \mathbb{W}$ (G-module splitting)

Identify π with projection onto $\mathbb{W}$
$v = (v_\pi, w)\ v_\pi \neq 0$

$(*)$ is equivalent to

$$(**) \qquad \overline{G \cdot [(v_\pi, w)]} \cap \overline{G \cdot [(v_\pi, 0)]} = \emptyset$$

(Zariski closure inside $\mathbb{P}(\mathrm{ker}(\pi) \oplus \mathbb{W}\ )$)

Given $(v\ ,\ w) \in \mathbb{V} \oplus \mathbb{W}$ set

$$\mathcal{O}_{vw} := G \cdot [(v, w)] \subset \mathbb{P}(\mathbb{V} \oplus \mathbb{W})$$

$$\mathcal{O}_{v} := G \cdot [(v, 0)] \subset \mathbb{P}(\mathbb{V} \oplus \{0\})$$

This motivates:

Definition . (Paul 2010) The pair (v, w) is ***semistable*** if and only if

$$\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_{v} = \emptyset$$

Example. Let $\mathbb{V}_e$ and $\mathbb{V}_d$ be irreducible $SL(2,\mathbb{C})$ modules with highest weights $e, d \in \mathbb{N} \cong$ homogeneous polynomials in two variables. Let f and g in $\mathbb{V}_e \setminus \{0\}$ and $\mathbb{W}_d \setminus \{0\}$ respectively.

Claim. (f, g) is semistable if and only if

$$e \leq d \text{ and for all } p \in \mathbb{P}^1 \ \mathrm{ord}_p(g) - \mathrm{ord}_p(f) \leq \frac{d-e}{2} \ .$$

When $e = 0$ and $f = 1$ conclude that $(1, g)$ is semistable if and only if

$$\mathrm{ord}_p(g) \leq \frac{d}{2} \text{ for all } p \in \mathbb{P}^1 \ .$$

Toric Morphisms

If the pair (v, w) is semistable then we certainly have that

$$\overline{T \cdot [(v, w)]} \cap \overline{T \cdot [(v, 0)]} = \emptyset$$

for all maximal algebraic tori $T \leq G$. Therefore there exists a morphism of ***projective*** toric varieties.

$$\begin{tikzcd}
 & \overline{T \cdot [(v, w)]} \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{P}(\mathbb{V} \oplus \mathbb{W}) \arrow[d, dashed, "\pi"] \\
T \arrow[ur] \arrow[dr] & & \\
 & \overline{T \cdot [(0, w)]} \arrow[r, hook] & \mathbb{P}(\mathbb{W})
\end{tikzcd}$$

We expect that the existence of such a morphism is completely dictated by the ***weight polyhedra*** : $\mathcal{N}(v)$ and $\mathcal{N}(w)$.

Theorem . (Paul 2012)
The following statements are equivalent.

1. (v, w) is ***semistable***. Recall that this means

$$\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_v = \emptyset$$

2. $\mathcal{N}(v) \subset \mathcal{N}(w)$ for all maximal tori $H \leq G$.
   We say that (v, w) is ***numerically semistable***.

3. For every maximal algebraic torus $H \leq G$ and $\chi \in \mathscr{A}_H(v)$ there exists an integer $d > 0$ and a ***relative invariant*** $f \in \mathbb{C}_d[\ \mathbb{V} \oplus \mathbb{W}\ ]^H_{d\chi}$ such that

$$f(v\ ,\ w) \neq 0 \text{ and } f|_{\mathbb{V}} \equiv 0\ .$$

Corollary A. If $\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_v \neq \emptyset$ then there exists an alg. 1psg $\lambda \in \Delta(G)$ such that

$$\lim_{\alpha \longrightarrow 0} \lambda(\alpha) \cdot [(v, w)] \in \overline{\mathcal{O}}_v \ .$$

Equip $\mathbb{V}$ and $\mathbb{W}$ with Hermitian norms . The ***energy of the pair*** (v, w) is the function on G defined by

$$G \ni \sigma \longrightarrow \mathsf{p}_{vw}(\sigma) := \log ||\sigma \cdot w||^2 - \log ||\sigma \cdot v||^2 \ .$$

Corollary B.

$$\inf_{\sigma \in G} \mathsf{p}_{vw}(\sigma) = -\infty$$

if and only if there is a degeneration $\lambda \in \Delta(G)$ such that

$$\lim_{\alpha \longrightarrow 0} \mathsf{p}_{vw}(\lambda(\alpha)) = -\infty \ .$$

Hilbert-Mumford Semistability	Semistable Pairs
For all $H \leq G$ $\exists$ $d \in \mathbb{Z}_{>0}$ and $f \in \mathbb{C}_{\leq d}[\,\mathbb{W}\,]^{H}$ such that $f(w) \neq 0$ and $f(0) = 0$	For all $H \leq G$ and $\chi \in \mathscr{A}_{H}(v)$ $\exists$ $d \in \mathbb{Z}_{>0}$ and $f \in \mathbb{C}_{d}[\,\mathbb{V} \oplus \mathbb{W}\,]^{H}_{d\chi}$ such that $f(v,w) \neq 0$ and $f\|_{\mathbb{V}} \equiv 0$
$0 \notin \overline{G \cdot w}$	$\overline{\mathcal{O}}_{vw} \cap \overline{\mathcal{O}}_{v} = \emptyset$
$w_{\lambda}(w) \leq 0$ for all 1psg's λ of G	$w_{\lambda}(w) - w_{\lambda}(v) \leq 0$ for all 1psg's λ of G
$0 \in \mathcal{N}(w)$ all $H \leq G$	$\mathcal{N}(v) \subset \mathcal{N}(w)$ all $H \leq G$
$\exists$ $C \geq 0$ such that $\log \|\|\sigma \cdot w\|\|^{2} \geq -C$ all $\sigma \in G$	$\exists$ $C \geq 0$ such that $\log \|\|\sigma \cdot w\|\|^{2} - \log \|\|\sigma \cdot v\|\|^{2} \geq -C$ all $\sigma \in G$

To summarize, the context for the study of ***SEMISTABLE PAIRS*** is

1. A reductive linear algebraic group G .

2. A pair $\mathbb{V}$, $\mathbb{W}$ of finite dimensional rational G-modules.

3. A pair of (non-zero) vectors $(v\ ,\ w) \in \mathbb{V} \oplus \mathbb{W}$.

Resultants and Discriminants

Let X be a smooth linearly normal variety

$$X \longrightarrow \mathbb{P}^N$$

Consider two polynomials:

$R_X :=$ ***X-resultant***

$\Delta_{X \times \mathbb{P}^{n-1}} :=$ ***X-hyperdiscriminant***

Let's *normalize the degrees* of these polynomials

$$X \to R = R(X) := R_X^{\deg(\Delta_{X \times \mathbb{P}^{n-1}})}$$

$$X \to \Delta = \Delta(X) := \Delta_{X \times \mathbb{P}^{n-1}}^{\deg(R_X)}$$

It is known that

$$R(X) \in \mathbb{E}_{\lambda_\bullet} \setminus \{0\}\ ,\ (n+1)\lambda_\bullet = \Big(\overbrace{r, r, \ldots, r}^{n+1}, \overbrace{0, \ldots, 0}^{N-n}\Big)\ .$$

$$\Delta(X) \in \mathbb{E}_{\mu_\bullet} \setminus \{0\}\ ,\ n\mu_\bullet = \Big(\overbrace{r, r, \ldots, r}^{n}, \overbrace{0, \ldots, 0}^{N+1-n}\Big)\ .$$

$$r = \deg(R(X)) = \deg(\Delta(X))\ .$$

$\mathbb{E}_{\lambda_\bullet}$ and $\mathbb{E}_{\mu_\bullet}$ are irreducible G ***modules***.

The associations $X \longrightarrow R(X)\ ,\ X \longrightarrow \Delta(X)$ are G equivariant:

$$R(\sigma \cdot X) = \sigma \cdot R(X)$$

$$\Delta(\sigma \cdot X) = \sigma \cdot \Delta(X)\ .$$

K-Energy maps and Semistable Pairs

Let P be a numerical polynomial

$$P(T) = c_n \binom{T}{n} + c_{n-1}\binom{T}{n-1} + O(T^{n-2}) \qquad c_n \in \mathbb{Z}_{>0} \ .$$

Consider the Hilbert scheme

$$\mathscr{H}^P_{\mathbb{P}^N} := \{ \text{ all (smooth) } X \subset \mathbb{P}^N \text{ with Hilbert polynomial } P \} \ .$$

Recall the G-equivariant morphisms

$$R \ , \ \Delta : \mathscr{H}^P_{\mathbb{P}^N} \longrightarrow \mathbb{P}(\mathbb{E}_{\lambda_\bullet}) \ , \ \mathbb{P}(\mathbb{E}_{\mu_\bullet}) \ .$$

Theorem (Paul 2012)

There is a constant M *depending only on* c_n, c_{n-1} *and the Fubini Study metric such that for all* $[X] \in \mathscr{H}^{P}_{\mathbb{P}^N}$ *and all* $\sigma \in G$ *we have*

$$|\,\nu_{\omega_{FS}|_X}(\varphi_\sigma) - \mathsf{p}_{R(X)\Delta(X)}(\sigma)\,| \leq M\ .$$