Ruijsenaars type deformation of hyperbolic BC_n Sutherland model

Ian Marshall

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History

1. Olshanetsky and Perelomov discovered the hyperbolic BC_n Sutherland model by a reduction/projection procedure, but it had only 2 independent parameters
2. Inozemtsev and Meshcheryakov proved integrability of the three parameter version
3. Feher-Pusztai obtained it with all three parameters by reduction: Start with $M = T^*K$ and reduce by canonical left and right actions of subgroups $K_+ \prec K$, with $K = SU(n,n)$ and K_+ the compact subgroup satisfying $gg^\dagger = I$: J^r is a character, and J^l is an appropriate analog of the so-called Kazhdan-Kostant-Sternberg element.

$$2H = \sum_{i=1}^{n} p_i^2 + a \sum_{i=1}^{n} \sinh^{-2}(2q_i) + b \sum_{i=1}^{n} \sinh^{-2}(q_i) + c \sum_{i,j \; i\neq j} \left[\sinh^{-2}(q_i + q_j) + \sinh^{-2}(q_i - q_j)\right].$$

Relativistic versions of Calogero type models were introduced by Ruijsenaars. It was proposed in several papers of Gorsky and others that Poisson Lie group reduction should be the appropriate setting for these systems.
Feher and Klimcik worked on this project and found PLG reduction interpretations for several known Ruijsenaars type systems.
My result is a PLG reduction construction of the same kind. It imitates the result of Feher and Pusztai, essentially amounting to the replacement of T^*K by the Heisenberg double of K. The product of this procedure is a *new* integrable system.
This is roughly speaking the whole story. The rest is a lot of tricky computations.

Reduction

Suppose we have a group G acting on a differentiable manifold M. The conerstone of reduction is the *decision to restrict from the ring $C^\infty(M)$ of all smooth functions on M to the subring of invariant functions $C^\infty(M)^G$.*
This means that we shall be interested only in invariant flows.
That is, for $v \in Vect(M)$, for any $\varphi \in C^\infty(M)$ we require

$$v(\varphi \circ g)(x) = v(\varphi)(g \cdot x), \qquad \text{where} \quad (\varphi \circ g)(x) = \varphi(g \cdot x). \quad (1)$$

With

$$\hat{\xi}(\varphi)(x) := \left.\frac{d}{dt}\right|_{t=0} \varphi\left(e^{t\xi} \cdot x\right),$$

then infinitesimally, (1) amounts to $\mathcal{L}_{\hat{\xi}} v = 0 \quad \forall \xi \in \mathfrak{g}$. Interpret v as a vector field on M/G, by restricting it to act only on $C^\infty(M)^G$.

Poisson case : M, G

Suppose now that M is a Poisson space, and that G is a Lie group acting on M in such a way that the Poisson structure is preserved. That is,

$$F, K \in C^\infty(M)^G \;\Rightarrow\; \{F, K\} \in C^\infty(M)^G. \tag{2}$$

In this case, $H \in C^\infty(M)^G$ is a sufficient condition for ensuring that the corresponding Hamiltonian vector field $\mathbb{X}_H \in Vect(M)$ is invariant.

Poisson reduction I : $\mathcal{F}$ and $\mathcal{F}_{gen}(\nu)$

Suppose that $\mathcal{F} \subset C^\infty(M)$ is a family of functions, all of whose Hamiltonian vector fields are tangent to the vector fields in $\hat{\mathfrak{g}}$, i.e., $\varphi \in \mathcal{F}, F \in C^\infty(M)^G \;\Rightarrow\; 0 = \mathbb{X}_\varphi(F) = -\mathbb{X}_F(\varphi)$.
Restricting to G-invariant functions, all the functions in $\mathcal{F}$ may be viewed as constants :
(i) Introduce $\mathcal{F}_{gen} := \{\varphi_\alpha \mid \alpha \in \mathcal{A}\}$, for which $\mathcal{F} = \langle \mathcal{F}_{gen} \rangle$.
(ii) For any set $\nu := \{\nu_\alpha \in \mathbb{R} \mid \alpha \in \mathcal{A}\}$, define

$$\mathcal{F}_{gen}(\nu) := \{\varphi_\alpha - \nu_\alpha \mid \alpha \in \mathcal{A}\}$$

(iii) Restrict to the submanifold

$$N_\nu := \bigcap_{\psi \in \mathcal{F}_{gen}(\nu)} \psi^{-1}(0) = \bigcap_{\alpha \in \mathcal{A}} \{x \in M \mid \varphi_\alpha(x) = \nu_\alpha\}.$$

Poisson reduction II : factor by G_ν

Suppose that $G_\nu \subset G$ is the maximal subgroup which acts on N_ν.

$$C^\infty(M)|_{N_\nu} \supset C^\infty(M)^G|_{N_\nu} = \left(C^\infty(M)|_{N_\nu}\right)^{G_\nu}$$

We arrive at a Poisson algebra consisting of the functions

$$C^\infty(N_\nu)^{G_\nu},$$

or, in other words, to the reduced space

$$N_\nu/G_\nu =: M_{red}(\nu) \quad \text{say.}$$

If $\mathbf{F} := \{F_i \mid i = 1, 2, \dots\}$ is a collection of invariant functions with the property $\{F_i, F_j\} = 0$, then their restrictions to N_ν will still have zero Poisson bracket, and they define commuting functions on the reduced space $M_{red}(\nu)$.

M : Heisenberg double, and $\mathcal{K}$: our symmetry group

Let $I_{nn} = \begin{pmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I_n} \end{pmatrix}$. G denotes $SL(2n, \mathbb{C})$.
K denotes $SU(n,n) = \{g \in SL(2n, \mathbb{C}) \mid g^\dagger I_{nn} g = I_{nn}\}$.
B denotes the set of all upper triangular matrices in $SL(2n, \mathbb{C})$ with real, positive diagonal entries, and B_n denotes the same set in $GL(n, \mathbb{C})$.

$K \supset K_+ = \left\{ \begin{pmatrix} p & \mathbf{0} \\ \mathbf{0} & q \end{pmatrix} \right\}, \quad p, q, \in U(n), \qquad \mathcal{K} = K_+ \times K_+$ acts on G by ordinary left and right multiplication.

The diagonal subgroup in $U(n)$ will be denoted by $\mathbb{T}$. For any $k \in K$ we may write

$$k = \begin{pmatrix} \rho & \mathbf{0} \\ \mathbf{0} & m \end{pmatrix} \begin{pmatrix} \Gamma & \Sigma \\ \Sigma & \Gamma \end{pmatrix} \begin{pmatrix} p & \mathbf{0} \\ \mathbf{0} & q \end{pmatrix}, \quad \text{with } \rho, m, p, q \in SU(n),$$

.

$\Gamma = diag(\cosh(\Delta_i))$, $\Sigma = diag(\sinh(\Delta_i))$

$\mathfrak{g} = Lie(G)$ can be decomposed as the sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$ of the two subalgebras $\mathfrak{k} = Lie(K)$ and $\mathfrak{b} = Lie(B)$, with respect to which the projections $P_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ and $P_{\mathfrak{b}} : \mathfrak{g} \to \mathfrak{b}$ are well-defined. Let $\langle \ , \ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote the non-degenerate, invariant inner product defined by

$$\langle X, Y \rangle = \operatorname{Im} \operatorname{tr} XY.$$

Then $R := P_{\mathfrak{k}} - P_{\mathfrak{b}}$ defines a classical r-matrix on $\mathfrak{g}$, skew-symmetric with respect to $\langle \ , \ \rangle$. For any function $F \in C^\infty(G)$, the left- and right-derivatives, $D^{l,r}F : G \to \mathfrak{g} \sim \mathfrak{g}^*$, of F are defined by

$$\left.\frac{d}{dt}\right|_{t=0} F(e^{tX} g e^{tY}) = \langle D^l F(g), X \rangle + \langle D^r F(g), Y \rangle \qquad \forall X, Y \in \mathfrak{g}.$$

The Poisson structure on G, viewed as the Heisenberg double based on the bi-algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$, is defined by

$$\{F, H\} = \langle D^l F, R(D^l H) \rangle + \langle D^r F, R(D^r H) \rangle.$$

Let $c_1, c_2 \in G$ and define the subspace $M(c_1, c_2) \subset G$ by

$$M(c_1, c_2) = \{bc_1k \mid b \in B, k \in K\} \cap \{kc_2b \mid b \in B, k \in K\}.$$

Introduce "coordinates" (b_L, k_L, b_R, k_R) on $M(c_1, c_2)$ (not independent) by

$$M(c_1, c_2) \ni g = b_L c_1 k_R = k_L c_2 b_R.$$

Proposition (Alekseev and Malkin – CMP1994)
$M(c_1, c_2)$ is a symplectic leaf, and all symplectic leaves are of this form. The symplectic structure on $M(c_1, c_2)$ can be written

$$[Symp](g) = \langle db_L b_L^{-1} \stackrel{\wedge}{,} dk_L k_L^{-1} \rangle \; + \; \langle b_R^{-1} db_R \stackrel{\wedge}{,} k_R^{-1} dk_R \rangle.$$

Constraints

Proposition:
The functions $\Phi_k \in C^\infty(G)$ defined by

$$\Phi_k(g) = -\frac{1}{2k}\mathrm{tr}\left(gI_{n,n}g^\dagger I_{n,n}\right)^k$$

are in involution with one another.
Constraints. Fixing $\sigma \in B_n$ and $x, y \in \mathbb{R}_+$, constraints are as follows: suppose that when written in the form
$G \ni g = k_L b_R,\ b_R = \begin{pmatrix} x\mathbf{I} & \omega \\ \mathbf{0} & x^{-1}\mathbf{I} \end{pmatrix}$
and that, when written in the form
$g = b_L k_R,\ b_L = \begin{pmatrix} y^{-1}\sigma & y^{-1}\nu \\ \mathbf{0} & y\mathbf{I} \end{pmatrix}$,
with $\det(\sigma) = 1$, and with both ω and ν undetermined in $gl(n)$.
$\sigma \in B_n$ is the PLG analog of the KKS element.

For $\xi, \eta \in \mathfrak{k}_+$, define

$$\begin{aligned} F_{\xi,\eta}(g) =& \mathrm{Im}\ \mathrm{tr}\, \xi \Bigg[g I_{nn} g^\dagger \ + \ y^{-2} g I_{nn} g^\dagger \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix} g I_{nn} g^\dagger \\ & -y^{-2} \begin{pmatrix} \sigma\sigma^\dagger & 0 \\ 0 & 0 \end{pmatrix} \Bigg] \\ & + \mathrm{Im}\ \mathrm{tr}\, \eta \Bigg[g^\dagger I_{nn} g - x^{-2} g^\dagger I_{nn} g \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix} g^\dagger I_{nn} g \\ & +x^{-2} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix} \Bigg], \end{aligned}$$

and then define the constraint submanifold by

$$N_{x,y,\sigma} := \{ g \in M(\mathbf{I}, \mathbf{I}) \mid F_{\xi,\eta}(g) = 0 \quad \forall \xi, \eta \in \mathfrak{k}_+ \}.$$

- back to the constraints

Factoring on the right by K_+ and on the left by the subgroup $\left\{\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & p\end{pmatrix}\right\} \subset K_+$, let's assume that g is in the gauge

$$g = \begin{pmatrix}\rho\Gamma & \rho\Sigma\\ \Sigma & \Gamma\end{pmatrix}\begin{pmatrix}x\mathbf{I} & \omega\\ \mathbf{0} & x^{-1}\mathbf{I}\end{pmatrix} \quad \text{with } \omega \in gl(n,\mathbb{C}),\ \rho \in SU(n)$$

Compare the two versions of $gI_{nn}g^\dagger$ and get constraint condition, for $T \in U(n)$

$$T^\dagger\Sigma^2 T = \Sigma\rho^\dagger\sigma\sigma^\dagger\rho\Sigma$$

Explanation of where we're up to:
Let us make the substitution $\Omega = \Sigma\omega + x^{-1}\Gamma$, so that

$$g = \begin{pmatrix} x\rho\Gamma & \rho\Sigma^{-1}(\Gamma\Omega - x^{-1}\mathbf{I}) \\ x\Sigma & \Omega \end{pmatrix}.$$

Now, from $gI_{nn}g^\dagger = b_L I_{nn} b_L^\dagger$ we get

$$\Omega\Omega^\dagger = y^2\mathbf{I} + x^2\Sigma^2 =: \Lambda^2 \;\Rightarrow\; \Omega = \Lambda T, \quad T \in U(n)$$

$$\nu = \rho\Sigma^{-1}(y^2\Gamma - x^{-1}\Omega^\dagger)$$

and

$$T^\dagger\Sigma^2 T = \Sigma\rho^\dagger\sigma\sigma^\dagger\rho\Sigma$$

The PLG KKS element is $\sigma \in B_n$, satisfying

$$\sigma\sigma^\dagger = \alpha^2 \mathbf{I} + \hat{v}\hat{v}^\dagger,$$

for $\alpha \in \mathbb{R}$ and $\hat{v} \in \mathbb{C}^n$ s.t. $\det \sigma = 1$, so the constraint reads

$$T^\dagger \Sigma^2 T = \alpha^2 \Sigma^2 + vv^T$$

$v := \Sigma\rho^\dagger \hat{v}$ and can be supposed real with all entries in $\mathbb{R}_{\geq 0}$.
The constraint condition can be solved(!)
After some work (quite a lot)

$$[Symp] = \langle\, \rho^\dagger d\rho \overset{\wedge}{,}\, \Sigma^{-1} T^\dagger \Sigma d(\Sigma^{-1} T \Sigma) \,\rangle \\ + \langle\, T^\dagger dT + dT T^\dagger \overset{\wedge}{,}\, \Sigma^{-1} d\Sigma \,\rangle.$$

After using a few more tricks, we arrive at

$$[Symp] = \sum_{i=1}^{n} dp_i \wedge \Sigma_i^{-1} d\Sigma_i$$

where, for $P := e^{ip} \in \mathbb{T}$, the general solution of the constraint condition was $T = P\tilde{T}$ and $\tilde{T}$ is a (complicated!) explicit, real matrix function of Σ.

The simplest of the commuting Hamiltonians produces

$$\Phi_1 = \tfrac{1}{2}(x^{-2} + y^2) \sum_{i=1}^{n} \Sigma_i^{-2}$$

$$- x^{-1} \sum_{i=1}^{n} (\cos p_i) \sqrt{1 + \Sigma_i^{-2}} \sqrt{x^2 + y^2 \Sigma_i^{-2}} \times$$

$$\prod_{k \neq i} \frac{\sqrt{\Sigma_k^2 - \alpha^2 \Sigma_i^2} \sqrt{\alpha^2 \Sigma_k^2 - \Sigma_i^2}}{\alpha(\Sigma_k^2 - \Sigma_i^2)}$$

The linearisation of the Hamiltonian - which means the degeneration of G to the cotangent bundle of T^*K, which is the same as the semi-direct product $K \ltimes \mathfrak{k}^* \sim K \ltimes \mathfrak{b}$ - yields the BC_n Hamiltonian of Feher and Pusztai

$$H_2 = \tfrac{1}{2}\sum_{i=1}^{n} \hat{p}_i^2 + c_1 \sum_{i=1}^{n} \frac{1}{\sinh^2 \hat{q}_i} + c_2 \sum_{i=1}^{n} \frac{1}{\sinh^2(2\hat{q}_i)} + c_3 \sum_{i,j\ \ i\neq j} \left[\frac{1}{\sinh^2(\hat{q}_i + \hat{q}_j)} + \frac{1}{\sinh^2(\hat{q}_i - \hat{q}_j)} \right].$$