The geometry of hydrodynamic integrability

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What is hydrodynamic integrability?

- A test for 'integrability' of 'dispersionless' systems of PDEs.
- Introduced by E. Ferapontov and K. Khusnutdinova [FeKh].
- Applies to systems which can be written in translation-invariant quasilinear first order form.
- 'Integrable' means system has sufficiently many 'hydrodynamic reductions' ($\Rightarrow$ Lots of solutions given by nonlinear superpositions of plane waves.)
- Known to be equivalent to integrability by dispersionless Lax pair in some cases [BFT,DFKN1,DFKN2,FHK].
- Computationally intensive: need symbolic computer algebra.

Quasilinear first order systems

A (translation-invariant first order) ***quasilinear system*** is a PDE system of the form [Tsa]

$$(1) \qquad A_1(\varphi)\partial_{x_1}\varphi + \cdots + A_n(\varphi)\partial_{x_n}\varphi = 0$$

on maps $\varphi\colon \mathbb{R}^n \to \mathbb{R}^s$, where $A_j\colon \mathbb{R}^s \to M_{k\times s}(\mathbb{R})$.

Example. An N-component hydrodynamic system is a system of the form

$$(2) \qquad \partial_{x_j} R_a = \mu_{aj}(R)\partial_{x_1} R_a$$

for $j \in \{2, \dots n\}$, $a \in \{1, \dots N\}$ and functions μ_{aj} of $R = (R_1, \dots R_N)$ which satisfy the compatibility conditions $\partial_b \mu_{aj} = \gamma_{ab}(R)(\mu_{bj} - \mu_{aj})$ for all $a \neq b$ and $j \in \{2, \dots n\}$.

An N-component ***hydrodynamic reduction*** of (1) is an ansatz $\varphi = F(R_1, \dots R_N)$ s.t. φ satisfies (1) if and only if R satisfies (2).

Example: dispersionless KP

Dispersionless limit of the Kadomtsev–Petviashvili equation:

$$\text{(dKP)} \qquad (u_t + uu_x)_x = u_{yy}.$$

Put into quasilinear first order form:

$$u_y - v_x = 0 = u_t + uu_x - v_y.$$

Substitute $u = U(R_1, \dots R_N)$ and $v = V(R_1, \dots R_N)$ with

$$\partial_t R_a = \lambda_a(R_1, \dots R_N)\partial_x R_a \quad \text{and} \quad \partial_y R_a = \mu_a(R_1, \dots R_N)\partial_x R_a,$$

using $u_x = \sum_a (\partial_a U)\partial_x R_a$ etc. to get

$$\sum_a (\mu_a \,\partial_a U - \partial_a V)\partial_x R_a = 0 = \sum_a \big((\lambda_a + U)\,\partial_a U - \mu_a \,\partial_a V\big)\partial_x R_a$$

so require $\mu_a \,\partial_a U = \partial_a V$ and $(\lambda_a + U)\,\partial_a U = \mu_a \,\partial_a V$ for all a.
In particular $\lambda_a + U = \mu_a^2$ (the ***dispersion relation***).

Method of hydrodynamic reductions

In general, the condition for functions $F(r_1, \dots r_N)$ and $\mu_{aj}(r_1, \dots r_N)$ to define a hydrodynamic reduction of a quasilinear system (1) is itself a PDE system.

For dKP, after eliminating V by $\partial_a V = \mu_a \, \partial_a U$ and $\lambda_a = \mu_a^2 - U$, the PDE system for U and μ_a to define a hydrodynamic reduction is that for all $a \neq b$,

$$\partial_b \mu_a = \frac{\partial_b U}{\mu_b - \mu_a}, \qquad \partial_a \partial_b U = 2 \frac{\partial_a U \, \partial_b U}{(\mu_b - \mu_a)^2},$$

Definition. A quasilinear system (1) is ***integrable by hydrodynamic reductions*** if the PDE system for N-component reductions is compatible for all $N \geq 2$.

(It then admits solutions depending on N functions of 1-variable.)

In fact the 2-component system is always compatible and it is enough to check $N = 3$ [FeKh].

Hydrodynamic integrability of dKP

In the dKP case, a tedious computation of the derivatives of the system yields

$$(3) \quad \partial_c(\partial_b\mu_a) = \frac{\partial_b U\,\partial_c U\,(\mu_b+\mu_c-2\mu_a)}{(\mu_b-\mu_c)^2(\mu_b-\mu_a)(\mu_c-\mu_a)},$$

$$(4) \quad \partial_c(\partial_a\partial_b U) = 4\frac{\partial_a U\,\partial_b U\,\partial_c U\,((\mu_a)^2+(\mu_b)^2+(\mu_c)^2-\mu_a\mu_b-\mu_a\mu_c-\mu_b\mu_c)}{(\mu_a-\mu_b)^2(\mu_a-\mu_c)^2(\mu_b-\mu_c)^2},$$

for all distinct a, b, c.

Since the RHS of (3) is symmetric in b, c and the RHS of (4) is totally symmetric in a, b, c, the system is compatible.

This is how the method works for one specific PDE with just one quadratic nonlinearity.

For anything remotely general, the computations are brutal.

What is going on?

Two clues to some underlying geometric meaning.

- The dispersion relation. For dKP, this says $[z_0, z_1, z_2] = [1, \lambda_a, \mu_a]$ is a point on $z_0 z_1 + u z_0^2 = z_2^2$.

  This quadric is the ***characteristic variety*** of dKP.

  For general hydrodynamic reductions, the ***characteristic momenta*** $\omega_a = \sum_{j=1}^{n} \mu_{aj}\, \mathrm{d}x_j$ are on the characteristic variety.
- Papers [BFT,DFKN1,DFKN2,FHK] showing that for three particular classes of systems, hydrodynamic reductions are nice submanifolds with respect to some interesting geometric structure on the codomain of φ.

Also inspiring ideas of A. Smith [Smi1,Smi2].

Plan for rest of talk

- Explain geometry of hydrodynamic reductions using the 'characteristic correspondence' of a quasilinear system.
- Use some algebraic geometry (projective embeddings) and differential geometry (nets) to give a fairly general result which unifies aforementioned observations of [BFT,DFKN1,DFKN2,FHK].

(But no progress yet on the harder, computationally intensive parts of these papers e.g. showing equivalence of hydrodynamic and Lax integrability.)

Quasilinear systems revisited

Natural context for quasilinear systems (QLS):

- Maps $\varphi\colon M \to \Sigma$ where M is an affine space modelled on an n-dimensional vector space $\mathfrak{t}$ and Σ is an s-manifold.
- Have $\mathrm{d}\varphi = \langle \psi, \mathrm{d}x \rangle \in \Omega^1(M, \varphi^* T\Sigma)$ where
  - $\psi \in C^\infty(M, \mathfrak{t}^* \otimes \varphi^* T\Sigma)$ and
  - $\mathrm{d}x \in \Omega^1(M, \mathfrak{t})$ is the tautological isomorphism $TM \cong M \times \mathfrak{t}$.
- QLS is $\psi \in C^\infty(M, \varphi^*\Psi)$ for a vector subbundle $\Psi \leq \mathfrak{t}^* \otimes T\Sigma$ over Σ (locally defined as kernel of some $A\colon \mathfrak{t}^* \otimes T\Sigma \to \mathbb{R}^k$).

Hydrodynamic case: Σ has coordinates $r_a : a \in \mathcal{A} = \{1, \ldots s\}$ and functions $\mu_a\colon \Sigma \to \mathfrak{t}^*$ s.t. Ψ is spanned by $\mu_a \otimes \partial_{r_a} : a \in \mathcal{A}$.

Equivalently, setting $\omega_a = \langle \mu_a, \mathrm{d}x \rangle$, the 2-forms $\omega_a \wedge \mathrm{d}r_a$ pull back to zero by $(id, \varphi)\colon M \to M \times \Sigma$.

(A very simple exterior differential system whose compatibility condition is $\mathrm{d}\omega_a \wedge \mathrm{d}r_a = 0 \;\; \forall\, a \in \mathcal{A}$.)

The characteristic correspondence

Projective bundle $\mathrm{P}(\mathfrak{t}^* \otimes T\Sigma) \to \Sigma$ has subbundle $\mathcal{R}$ with fibre

$$\mathcal{R}_p := \{[\xi \otimes Z] : \xi \in \mathfrak{t}^*,\, Z \in T_p\Sigma\}$$

i.e., rank one tensors – ***Segre image*** of $\mathrm{P}(\mathfrak{t}^*) \times \mathrm{P}(T_p\Sigma)$.

Definition. Let $\Psi \leq \mathfrak{t}^* \otimes T\Sigma$ be a QLS.

- ***Rank one variety*** of Ψ is $\mathcal{R}^\Psi := \mathcal{R} \cap \mathrm{P}(\Psi)$.
- ***Characteristic*** and ***cocharacteristic varieties*** of Ψ are projections χ^Ψ and $\mathcal{C}^\Psi$ of $\mathcal{R}^\Psi$ to $\Sigma \times \mathrm{P}(\mathfrak{t}^*)$ and $\mathrm{P}(T\Sigma)$ resp.
- ***Characteristic correspondence*** of Ψ:

$$\begin{array}{ccc} & \mathcal{R}^\Psi & \\ {\scriptstyle \pi_\chi}\swarrow & & \searrow{\scriptstyle \pi_{\mathcal{C}}} \\ \Sigma \times \mathrm{P}(\mathfrak{t}^*) \geq \chi^\Psi & & \mathcal{C}^\Psi \leq \mathrm{P}(T\Sigma) \end{array}$$

(Assumed smooth double fibration.)

Examples

- Hydrodynamic system: $\chi^{\Psi} = \{[\mu_a] : a \in \mathcal{A}\}$, $\mathcal{C}^{\Psi} = \{[\partial_{r_a}] : a \in \mathcal{A}\}$, $\mathcal{R}^{\Psi} = \{[\mu_a \otimes \partial_{r_a}] : a \in \mathcal{A}\}$.
- dKP: $\varphi = (u, v)\colon M = \mathbb{R}^3 \to \Sigma = \mathbb{R}^2$. Then $\Psi_{(u,v)}$ is $\{(u_x, u_y, u_t) \otimes (1, 0) + (u_y, u_t + uu_x, v_t) \otimes (0, 1)\}$ and rank one elts have (u_x, u_y, u_t) and $(u_y, u_t + uu_x, v_t)$ lin. dep., giving
$$\mathcal{R}^{\Psi}_{(u,v)} = \{(\lambda^2, \lambda\mu, \mu^2 - u\lambda^2) \otimes (\lambda, \mu) : \lambda, \mu \in \mathbb{R}\}.$$
Then $\mathcal{C}^{\Psi} = P^1$, and χ^{Ψ} is a u-dependent conic in P^2.
- For $\Sigma \subseteq \mathfrak{t}^* \otimes V \subseteq \mathrm{Gr}_n(\mathfrak{t} \oplus V)$, $\varphi\colon M \to \Sigma$ is derivative of $u\colon M \to V$ iff $\psi(x) \in \Psi_{\varphi(x)}$ with $\Psi_p := \mathfrak{t}^* \otimes T_p\Sigma \cap S^2\mathfrak{t}^* \otimes V \subseteq \mathfrak{t}^* \otimes \mathfrak{t}^* \otimes V$. Then
$$\chi^{\Psi}_p = \{[\xi] \in \mathrm{P}(\mathfrak{t}^*) : \xi \otimes v \in T_p\Sigma \text{ for some } v \in V\}$$
$$\mathcal{C}^{\Psi}_p = \{[\xi \otimes v] \in \mathrm{P}(T_p\Sigma)\}$$
$$\mathcal{R}^{\Psi}_p = \{[\xi \otimes \xi \otimes v] \in \mathrm{P}(\mathfrak{t}^* \otimes T_p\Sigma)\}.$$
Get many examples this way (including Ferapontov et al.).

Hydrodynamic reductions revisited

Seek to write $\varphi = S \circ R$ with $R\colon M \to \mathbb{R}^N$ and $S\colon \mathbb{R}^N \to \Sigma$ so that φ solves Ψ iff $\forall\, a \in \mathcal{A} = \{1, \dots N\}, \quad \mathrm{d}R_a \wedge \langle \mu_a(R), \mathrm{d}x\rangle = 0$ i.e., $\mathrm{d}R_a = f_a(R)\langle \mu_a(R), \mathrm{d}x\rangle$ for some functions f_a.

Chain rule: $$\mathrm{d}\varphi = R^*\mathrm{d}S \circ \mathrm{d}R = \sum_{a\in\mathcal{A}} \mathrm{d}R_a \otimes \partial_a S(R)$$

$$= \sum_{a\in\mathcal{A}} f_a(R)\langle \mu_a(R), \mathrm{d}x\rangle \otimes \partial_a S(R) = \langle \psi, \mathrm{d}x\rangle,$$

where $$\psi = \sum_{a\in\mathcal{A}} f_a(R)\mu_a(R) \otimes \partial_a S(R).$$

Want many solns: $\mu_a \otimes \partial_a S \in \Psi$

Definition. An N-***component hydrodynamic reduction*** of a QLS $\Psi \leq \mathfrak{t}^* \otimes T\Sigma$ is a map

$$(S, [\mu_1], \dots [\mu_N])\colon \mathbb{R}^N \to \chi^{\Psi} \times_\Sigma \cdots \times_\Sigma \chi^{\Psi}$$

(N-fold fibre product) s.t. $\mu_a \otimes \partial_a S$ is in Ψ for all a, and the hydrodynamic system defined by μ_a is compatible.

Main result

So far: turned simple-minded but fearsome calculus into abstract nonsense geometry. No PDE person would call this progress.

So do we win anything?

Theorem. Let $\Psi \leq \mathfrak{t}^* \otimes T\Sigma$ be a compliant QLS. Then modulo natural equivalences, generic N-component hydrodynamic reductions of Ψ, with $N \leq \dim \Sigma$, correspond bijectively to N-dimensional cocharacteristic nets in Σ.

Remaining business:

- Explain what is a ***compliant*** QLS (alg. geom.)
- Explain what is a ***cocharacteristic net*** (diff. geom.)
- Prove the theorem

Algebraic geometry: projective embeddings

- χ^{Ψ} and $\mathcal{C}^{\Psi}$ are fibrewise projective varieties in projectivized vector bundles, and the corresponding dual tautological line bundles pull back to line bundles $L_{\chi} \to \chi^{\Psi}$ and $L_{\mathcal{C}} \to \mathcal{C}^{\Psi}$.
- For a line bundle L over a bundle of projective varieties over Σ, let $H^0(L) \to \Sigma$ be the bundle of fibrewise regular sections.
- Have canonical maps $\Sigma \times \mathfrak{t} \to H^0(L_{\chi})$ and $T^*\Sigma \to H^0(L_{\mathcal{C}})$ given by restricting fibrewise sections of the dual tautological line bundles to χ^{Ψ} and $\mathcal{C}^{\Psi}$.
- If χ^{Ψ} and $\mathcal{C}^{\Psi}$ are not contained (fibrewise) in any hyperplane, these maps are injective, hence fibrewise linear systems, and surjectivity means that these linear systems are complete.

Compliant QLS

A QLS is ***compliant*** if the following conditions hold:

1. the characteristic correspondence maps are isomorphisms, and we let $\zeta^\Psi = \pi_\chi \circ \pi_{\mathcal{C}}^{-1}$ be the induced isomorphism $\mathcal{C}^\Psi \to \chi^\Psi$;
2. the canonical maps $\Sigma \times \mathfrak{t} \to H^0(L_\chi)$ and $T^*\Sigma \to H^0(L_{\mathcal{C}})$ are isomorphisms;
3. $\mathcal{V}^\Psi := H^0(L_{\mathcal{C}} \otimes (\zeta^\Psi)^* L_\chi^*)^* \to \Sigma$ is a nonzero vector bundle, and the canonical vector bundle map $T\Sigma \to \mathfrak{t}^* \otimes \mathcal{V}^\Psi$—induced by the transpose of the tensor product map
$$H^0((\zeta^\Psi)^* L_\chi) \otimes H^0(L_{\mathcal{C}} \otimes (\zeta^\Psi)^* L_\chi^*) \to H^0(L_{\mathcal{C}})$$
—is an embedding;
4. if $\mathrm{rank}(\mathcal{V}^\Psi) \geq 2$, no 2-dimensional submanifold of Σ has rank one tangent space in $\mathfrak{t}^* \otimes \mathcal{V}^\Psi$.

Key point: under isomorphism in 1., $L_{\mathcal{C}}$ is at least as ample as L_χ by 3., so $T\Sigma$ has a tensor product decomposition using 2.

Differential geometry: nets

- A ***pre-net*** on an N-manifold Q is a direct sum decomposition $TQ = \bigoplus_{j\in\mathcal{J}} \mathcal{D}_j$ into rank one distributions $\mathcal{D}_j \leq TQ$ for $j \in \mathcal{J} := \{1, \dots N\}$.
- A pre-net $\mathcal{D}_j : j \in \mathcal{J}$ on Q is ***integrable*** if for every subset $\mathcal{I} \subseteq \mathcal{J}$, $\mathcal{D}_{\mathcal{I}} := \bigoplus_{i\in\mathcal{I}} \mathcal{D}_i$ is an integrable distribution (i.e., tangent to a foliation with $\#\mathcal{I}$ dimensional leaves); an integrable pre-net is called a ***net***.

Frobenius theorem gives characterizations of integrability.
Also need a special class of nets.

- If $\mathcal{D}_j : j \in \mathcal{J}$ is a pre-net on Q, and $TQ \leq \mathcal{V} \otimes \mathfrak{t}^*$ for a line bundle $\mathcal{V} \to Q$ and a vector space $\mathfrak{t}^*$, then each $\mathcal{D}_i$ defines a line subbundle M_i of $Q \times \mathfrak{t}^*$.
- May then require that for any section X_i of $\mathcal{D}_i$, have $\mathrm{d}_{X_i} M_j \leq M_i \oplus M_j$. If this holds then $\mathcal{D}_j : j \in \mathcal{J}$ is a net and will be called a ***conjugate net***.

(Well known when Q is an affine space with translation group $\mathfrak{t}$.)

Cocharacteristic nets

Let $\Psi \leq \mathfrak{t}^* \otimes T\Sigma$ be a compliant QLS with $T\Sigma \leq \mathfrak{t}^* \otimes \mathcal{V}^\Psi$.

An N-dimensional ***cocharacteristic net*** in Σ is an N-dimensional submanifold $S\colon \mathbb{R}^N \to \Sigma$ such that:

1. the net spanned by $\partial_a S : a \in \mathcal{A}$ satisfies $[\partial_a S] \in \mathcal{C}^\Psi$; and
2. if $\mathcal{V}^\Psi$ has rank one, the net is conjugate.

Clearly a hydrodynamic reduction defines a net satisfying 1.

Conversely, given such a net, the embedding of $\mathcal{C}^\Psi$ into $\mathrm{P}(\mathfrak{t}^* \otimes \mathcal{V}^\Psi)$ gives $\partial_a S = \mu_a \otimes v_a$ for some local sections v_a of $S^*\mathcal{V}^\Psi$.

The main point is to show that the compatibility of the hydrodynamic system with characteristic momenta $\langle \mu_a, \mathrm{d}x \rangle$ is equivalent to 2.

Proof of Theorem

Choose a basis for $\mathfrak{t}^*$ and rescale characteristic momenta s.t. $\mu_{a1} = 1$. Then have $\partial_b S_k = \mu_{bk} \partial_b S_1 = \mu_{bk} v_b$ for $k \in \{1, \dots n\}$.

Differentiate by ∂_a and commute partial derivatives to obtain

$$(\partial_a \mu_{bk})\partial_b S_1 - (\partial_b \mu_{ak})\partial_a S_1 = (\mu_{ak} - \mu_{bk})\partial_a \partial_b S_1. \tag{5}$$

Dividing by $\mu_{ak} - \mu_{bk}$, RHS is independent of k so

$$\Big(\frac{\partial_a \mu_{bk}}{\mu_{ak} - \mu_{bk}} - \frac{\partial_a \mu_{b\ell}}{\mu_{a\ell} - \mu_{b\ell}}\Big) v_b = \Big(\frac{\partial_b \mu_{ak}}{\mu_{ak} - \mu_{bk}} - \frac{\partial_b \mu_{a\ell}}{\mu_{a\ell} - \mu_{b\ell}}\Big) v_a.$$

Both sides are zero unless v_a and v_b are lin. dep., i.e., multiples of some $v \in \mathcal{V}^\Psi$, say. But then span of $\partial_a S = \mu_a \otimes v_a$ and $\partial_b S = \mu_b \otimes v_b$ is $\mathrm{span}\{\mu_a, \mu_b\} \otimes \mathrm{span}\{v\}$, i.e., entirely rank one.

For $\mathrm{rank}(\mathcal{V}^\Psi) > 1$, the set where this holds has empty interior by compliancy, so hydrodynamic compatibility condition is satisfied on dense complement, hence everywhere by continuity.

The rank one case

If $\partial_a \mu_{bk} = \gamma_{ba}(\mu_{ak} - \mu_{bk})$ for $a \neq b$, have

$$\begin{aligned}
\partial_a \partial_b S_k &= (\partial_a \mu_{bk}) \partial_b S_1 + \mu_{bk}\, \partial_a \partial_b S_1 \\
&= \gamma_{ba}(\mu_{ak} - \mu_{bk}) \partial_b S_1 + \mu_{bk}(\gamma_{ab} \partial_a S_1 + \gamma_{ba} \partial_b S_1) \\
&= \gamma_{ab}(v_a / v_b) \partial_b S_k + \gamma_{ba}(v_b / v_a) \partial_a S_k
\end{aligned}$$

by (5) so S is conjugate.

Conversely, if S is conjugate with $\partial_a \partial_b S_k = \alpha_{ab} \partial_b S_k + \beta_{ab} \partial_a S_k$ for $a \neq b$, then taking $k = 1$, have

$$\partial_a \partial_b S_1 = \alpha_{ab} \partial_b S_1 + \beta_{ab} \partial_a S_1 = \alpha_{ab} v_b + \beta_{ab} v_a.$$

On the other hand

$$\mu_{bk}\, \partial_a \partial_b S_1 = \partial_a \partial_b S_k - (\partial_a \mu_{bk}) \partial_b S_1 = \alpha_{ab} \mu_{bk} v_b + \beta_{ab} \mu_{ak} v_a - (\partial_a \mu_{bk}) v_b$$

Now eliminate $\partial_a \partial_b S_1$ to obtain

$$\alpha_{ab} \mu_{bk} v_b + \beta_{ab} \mu_{bk} v_a = \alpha_{ab} \mu_{bk} v_b + \beta_{ab} \mu_{ak} v_a - (\partial_a \mu_{bk}) v_b$$

and hence $\partial_a \mu_{bk} = \beta_{ab}(v_a / v_b)(\mu_{ak} - \mu_{bk})$. □

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