Reflection groups and q-reflection groups

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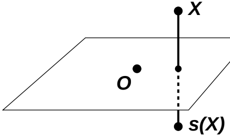
Reflections V = vector space over $\mathbb{k}$, dim V is finite.

$s \in GL(V)$ is a (pseudo)**reflection** if s is of finite order, $codim\, V^s = 1$.

- real reflections ($\mathbb{k} = \mathbb{R}$): $s \sim \mathrm{diag}(1, 1, \ldots, 1, -1)$



reflecting hyperplane $= ker(Id - s)$

- complex reflections ($\mathbb{k} = \mathbb{C}$): $s \sim \mathrm{diag}(1, 1, \ldots, 1, \varepsilon)$
  $\varepsilon \neq 1$ a root of 1
- char $\mathbb{k} > 0$: s may not be diagonalisable

Finite reflection groups (subgps of $GL(V)$ generated by reflections)

NB: Finiteness is a very strong condition!

Only very special arrangements of reflecting hyperplanes ("mirrors") lead to finite reflection groups.

Reflection groups over $\mathbb{Q}$ = **Weyl groups**
(extremely important in the theory of semisimple Lie algebras)

$$\bigcap$$

Real reflection groups = **Coxeter groups**

$$\bigcap$$

Complex reflection groups

Finite reflection groups: classification over $\mathbb{Q}$ and $\mathbb{R}$

A reflection group can be characterised by the set of $\pm$normals to mirrors (*roots*)

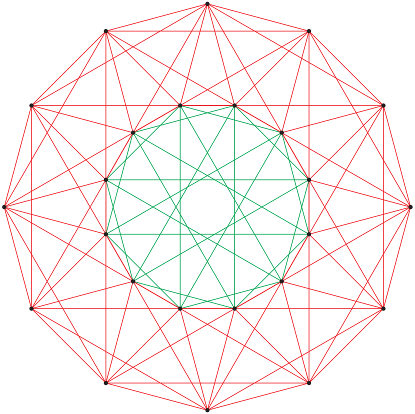
- For example: $\mathbb{R}^{n+1} \ni \{e_i - e_j : 1 \leq i \neq j \leq n+1\}$ reflections $s_{ij} : e_i \leftrightarrow e_j$ generate **symmetric group** S_{n+1} (Weyl group of type A_n, $n \geq 1$)

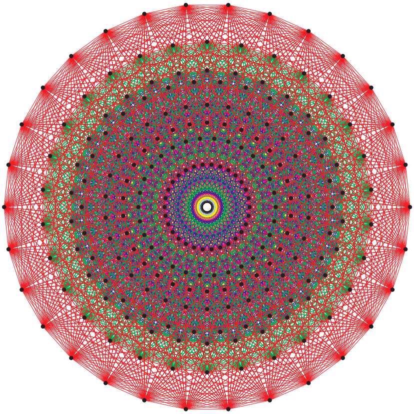
- Weyl group of type B_n, $n \geq 2$: $\mathbb{R}^n \ni \{\pm e_i \pm e_j : 1 \leq i \neq j \leq n\} \cup \{\pm e_i : 1 \leq i \leq n\}$ reflection-generators $s_{ij} : e_i \leftrightarrow e_j$, $t_i : e_i \leftrightarrow -e_i$ (hyperoctahedral group, order $2^n n!$)

- Also, D_n ($n \geq 4$), E_6, E_7, E_8, F_4, G_2 are Weyl groups $I_2(m)$, H_3, H_4 are “extra” Coxeter groups

Root systems of D_4 and E_8

(planar projection of the polytope which is the convex hull of the root system)





The Shephard – Todd classification of finite complex reflection groups (1954)

They all are direct products of the following groups:

- $G = G(m, p, n) \leq GL_n(\mathbb{C}), \qquad p|m$
  (invertible $n \times n$ matrices with exactly n non-zero entries which are mth roots of 1, their product is an (m/p)th root of 1)
- $G =$ one of the exceptional groups $G_4, \ldots, G_{37}$.

Notation: $S(V)^G = \{p \text{ in } S(V) \ : \ g(p) = p \quad \forall\, g \in G\}$

The Chevalley – Shephard – Todd theorem (1955)

Assume that $\mathrm{char}\, \Bbbk = 0$. A finite $G < GL(V)$ is a complex reflection group, if and only if $S(V)^G$ is a polynomial algebra.

Remark on generators of $S(V)^G$

$S(V)$ is an algebra of polynomials in $n = \dim V$ variables.

If $G < GL(V)$ is a finite complex reflection group, $S(V)^G$ has n algebraically independent generators $p_1, \dots, p_n$.

Moreover, $p_1, \dots, p_n$ may be chosen to be homogeneous.

$p_1, \dots, p_n$ are not unique, but $\{d_1, \dots, d_n\} = \{\deg p_1, \dots, \deg p_n\}$ is uniquely determined by G (*degrees of* G).

One has $d_1 d_2 \dots d_n = |G|$.

Example $G = \mathbb{S}_n$ symmetric group $\leq GL_n(\mathbb{C})$

$p_1, \dots, p_n$ are, e.g., elementary symmetric polynomials in n variables

Degrees: $d_1 = 1, d_2 = 2, \dots, d_n = n$

Generalisations of the C-S-T theorem

(1) $\mathrm{char}\ \Bbbk > 0$.

Serre (1970s) proved that if $S(V)^G$ is polynomial, then G is a reflection group, and for any proper subspace $W \subset V$, H=the stabiliser of W has polynomial $S(W)^H$.

Kemper, Malle (1997) proved "if and only if" (using a classification of pseudoreflection groups due to Kantor, Wagner, Zalesskii, Serezhin).

(2) Replace $S(V)$ with some noncommutative algebra, on which the group G acts.

(In other words, consider a "*noncommutative space*" with an action of G.)

Below is a particular case of this:

$V = \mathbb{C}$-span of $x_1, \ldots, x_n$; $\mathbf{q} = \{q_{ij}\}_{i,j=1}^n$, $q_{ii} = 1$, $q_{ij} q_{ji} = 1\ \forall\, i, j$

$S_{\mathbf{q}}(V) = \langle x_1, \ldots, x_n | x_i x_j = q_{ij} x_j x_i \rangle$ "the algebra of q-polynomials"

Problem 1: Find finite G such that G acts on $S_{\mathbf{q}}(V)$ and $S_{\mathbf{q}}(V)^G$ is also a q'-polynomial algebra.

("q-reflection groups"?)

B.-Berenstein, 2009:

instead of solving Problem 1, solved **a different problem** (Problem 2 below) such that:

- if $q_{ij} = 1 \; \forall \, i, j$ (the *commutative case*), the solution to **Problem 1** AND to **Problem 2** are reflection groups.

The semidirect product $S(V) \rtimes G$

To see what Problem 2 is about, condider the following.

Definition: The semidirect product $S(V) \rtimes G$ is the algebra generated by V and by the algebra $\mathbb{C}G$ subject to relations

$g \cdot v = g(v) \cdot g$ for $g \in G$, $v \in V$; $\quad [v_1, v_2] = 0 \ \forall v_1, v_2 \in V$.

Important property: if $x_1, \dots, x_n$ are a basis of V,

$$\{x_1^{k_1} \dots x_n^{k_n} g \mid k_i \in \mathbb{Z}_{\geq 0}, \ g \in G\}$$

is a basis of $S(V) \rtimes G$.

In other words, $S(V) \rtimes G$ is $S(V) \otimes \mathbb{C}G$ as a vector space.

Drinfeld (1985) suggested the following **deformation** of the defining relations of $S(V) \rtimes G$. Let A be the algebra generated by V and by the algebra $\mathbb{C}G$ subject to relations

$g \cdot v = g(v) \cdot g$ for $g \in G$, $v \in V$; $\qquad [v_1, v_2] = \sum_{g \in G} a_g(v_1, v_2) g$.

Here $a_g \colon V \times V \to \mathbb{C}$ are bilinear forms.

Clearly, the above set

$$\{x_1^{k_1} \ldots x_n^{k_n} g\} \qquad (\dagger)$$

of monomials spans A, but it may now be linearly dependent, and A may be "strictly smaller" than $S(V) \otimes \mathbb{C}G$.

The set $\{a_g : g \in G\} \subset (V \otimes V)^*$ is called **admissible**, if the monomials $(\dagger)$ are a basis of A.

- PBW-type basis
    - A is a flat deformation of $S(V) \rtimes G$

The following conditions are **necessary** for $\{a_g : g \in G\}$ to be admissible: for $v_i \in V$, $g \in G$,

- $[v_1, v_2] = -[v_2, v_1]$, so a_g is skew-symmetric;
- $g \cdot [v_1, v_2] = [g(v_1), g(v_2)] \cdot g$, so $a_h(v_1, v_2) = a_{ghg^{-1}}(g(v_1), g(v_2))$;
- $[[v_1, v_2], v_3] + [[v_2, v_3], v_1] + [[v_3, v_1], v_2] = 0$ (Jacobi identity), which rewrites as

$g \neq 1,\ a_g \neq 0 \qquad \Rightarrow \qquad \ker(a_g) = V^g$ and $codim(V^g) = 2$.
Here $V^g = \{v \in V : g(v) = v\}$.

Drinfeld claimed that the above conditions are **sufficient** for $\{a_g\}$ to be admissible. This claim is true.

Definition A, which is a flat deformation of $S(V) \rtimes G$, is called a degenerate affine Hecke algebra.

Problem 2(D): Find such A for a given $G < \mathrm{GL}(V)$. ([Dr'85]: $G = S_n$ or Coxeter gp.)

History

Q. Why study flat deformations of $S(V) \rtimes G$?

A. Representation theory, geometry (orbifolds V/G), Lie theory etc.

For example:

- Lusztig (1989) introduced the "graded affine Hecke algebra" of a Weyl group G, a deformation of the **semidirect product** relation in $S(V) \rtimes G$.
- Etingof, Ginzburg (2002) introduced the **symplectic reflection algebras** which are degenerate affine Hecke algebras for G which preserves a symplectic form ω on V.

(*Both were done without knowing about Drinfeld's earlier construction.*)

Particular case: The split symplectic case

$G < \mathrm{GL}(V)$, the algebra to be deformed is $S(V \oplus V^*) \rtimes G$.

There is always a non-trivial deformation, **the Heisenberg-Weyl algebra** $\mathcal{A}(V)$:

$$\forall x, x' \in V^*,\ v, v' \in V$$
$$[x, x'] = 0, \qquad [v, v'] = 0, \qquad [v, x] = \langle v, x\rangle \cdot 1,$$

where $\langle\ ,\ \rangle$ is the canonical pairing between V and V^*.

$\mathcal{A}(V)$ is the most straightforward quantisation of the phase space $V \oplus V^*$.

If $\langle \xi, x\rangle \cdot 1$ is replaced by an expression in $\mathbb{C}G$ and the deformation is still flat, one has a **rational Cherednik algebra** of G.

These are introduced and classified in [EG, Invent. Math., '02] and correspond to **complex reflection groups**.

Problem 2: Find finite G for which there is a q-analogue of the **rational Cherednik algebra** of G.

Dunkl operators

$\frac{\partial}{\partial v}$, $v \in V$, are commuting operators on $S(V^*)$.

NB: $\frac{\partial}{\partial v} p = [v, p]$ in the algebra $\mathcal{A}(V)$, where $p \in S(V^*)$.

<u>Deformation</u>: Replace $\mathcal{A}(V) \cong S(V \oplus V^*)$ with a rational Cherednik algebra $H_C(G) \cong S(V \oplus V^*) \otimes \mathbb{C}G$ of $G < \mathrm{GL}(V)$:

$$\boxed{\nabla_v p = \frac{\partial p}{\partial v} + \sum_s c_s \cdot \alpha_s(v) \cdot \frac{p - s(p)}{\alpha_s}}$$, where

- s runs over complex reflections in $G < GL(V)$
- c_s are scalar parameters such that $c_{gsg^{-1}} = c_s$ for all $g \in G$
- $\alpha_s \in V^*$ is the root of s: $s(v) = v - \alpha_s(v)\alpha_s^\vee$ for some $\alpha_s^\vee \in V$

These operators were first introduced by Dunkl (1989) for Coxeter groups (in harmonic analysis).

Dunkl operators commute

Theorem [Du,EG]: ∇_v(polynomials) $\subseteq$ polynomials, $\nabla_u \nabla_v = \nabla_v \nabla_u$

Proof (using rational Cherednik algebras): $H_C(G)$ acts on $S(V^*)$ via induced representation. The action of $v \in V$ is via the Dunkl operator ∇_v. But $v \in V$ commute in $H_C(G)$.

Example for $G = \mathbb{S}_n$:

$$\nabla_i = \frac{\partial}{\partial x_i} + c \sum_{j \neq i} \frac{1}{x_i - x_j}(1 - s_{ij})$$

$\nabla_1, \ldots, \nabla_n$ act on $\mathbb{C}[x_1, \ldots, x_n]$ and commute.

The rational Cherenik algebra is a flat deformation of

$\mathcal{A}(V) \rtimes G \cong S(V) \otimes \mathbb{C}G \otimes S(V^*)$ (triangular decomposition).

[EG] prove this, using the Koszul deformation principle.

[B.-Berenstein, Adv. Math. '09] introduce *braided doubles* (a more general class of algebras defined by triangular decomposition):

$T(V)/I^- \otimes H \otimes T(W)/I^+$ where V, W are modules over a Hopf algebra H, $I^\pm$ are two-sided ideals, $[V, W] \subset H$.

Example (the differential calculus on a noncommutative space):

Y is a space with a **braiding** $\Psi \in End(Y \otimes Y)$,

i.e., $(Id \otimes \Psi)(\Psi \otimes Id)(Id \otimes \Psi) = (\Psi \otimes Id)(Id \otimes \Psi)(\Psi \otimes Id)$.

$\rightsquigarrow$ **braided Weyl algebra** $\mathcal{A}(Y, \Psi) \cong \mathcal{B}(Y) \otimes \mathcal{B}(Y^*)$.

Here $\mathcal{B}(Y)$, $\mathcal{B}(Y^*)$ are **Nichols algebras** which have relations that depend on Ψ.

Anticommuting Dunkl operators

Theorem [B.-Berenstein] If $D = T(V)/I^- \otimes \Bbbk G \otimes T(W)/I^+$ is a minimal braided double, there exist a finite-dimensional braided space (Y, Ψ) so that D embeds in $\mathcal{A}(Y, \Psi) \rtimes G$.

Thus, one may look for algebras with triangular decomposition and with given relations among certain subalgebras of braided Weyl algebras $\mathcal{A}(Y, \Psi) \rtimes G$.

For example [B.-Berenstein, Selecta Math. '09]:

Let $\underline{x}_1, \ldots, \underline{x}_n$ be **anticommuting** variables, $\underline{x}_i \underline{x}_j = -\underline{x}_j \underline{x}_i$, $i \neq j$

Look for algebras of the form

$\mathbb{C}\langle \underline{v}_1, \ldots, \underline{v}_n \rangle \otimes \mathbb{C}G \otimes \mathbb{C}\langle \underline{x}_1, \ldots, \underline{x}_n \rangle$, $\underline{x}_i \underline{v}_j - (-1)^{\delta_{ij}} \underline{v}_j \underline{x}_i \in \mathbb{C}G$

Theorem 1 (Solution to Problem 2) The above algebras with triangular decomposition exist for, and only for, the following groups:

- $G = G(m, p, n)$, (m/p) even
- $G = G(m, p, n)_+$, (m/p) even, $(m/2p)$ odd

Definition For finite $G < GL(V)$, consider the character $\det\colon G \to \mathbb{C}^\times$ and put $\mathcal{C} = \det(G)$ (finite cyclic group). Then

$$G_+ = \{g \in G : \det(g) \in \mathcal{C}^2\}$$

(the subgroup of even elements of G).

(**NB** Either $G_+ = G$ or $|G : G_+| = 2$)

Smallest group in rank n: $G = G(2, 1, n)_+ =$ even elements in the Coxeter group of type B_n (denoted B_n^+)

Anticommuting Dunkl operators for B_n^+

$$\boxed{\underline{\nabla}_i = \underline{\partial}_i + c\sum_{j\neq i}\frac{x_i + x_j}{x_i^2 - x_j^2}(1-\sigma_{ij}) + \frac{x_i - x_j}{x_i^2 - x_j^2}(1-\sigma_{ji})},$$

$i = 1, \ldots, n$

- $\underline{\partial}_i$ are anticommuting skew-derivations of $\mathbb{C}\langle \underline{x}_1, \ldots, \underline{x}_n\rangle$
- σ_{ij} is an automorphism of $\mathbb{C}\langle \underline{x}_1, \ldots, \underline{x}_n\rangle$ of order 4,

$$\sigma_{ij}(\underline{x}_i) = \underline{x}_j, \qquad \sigma_{ij}(\underline{x}_j) = -\underline{x}_i, \qquad \sigma_{ij}(\underline{x}_k) = \underline{x}_k, \qquad k \neq i, j.$$

- **NB** $\underline{x}_i^2 - \underline{x}_j^2$ is central in $\mathbb{C}\langle \underline{x}_1, \ldots, \underline{x}_n\rangle$, division is well-defined; $\underline{x}_i^2 - \underline{x}_j^2 \neq (\underline{x}_i - \underline{x}_j)(\underline{x}_i + \underline{x}_j)$

Theorem 2 $\underline{\nabla}_i$(skew-polynomials) $\subseteq$ skew-polynomials, $\underline{\nabla}_i\underline{\nabla}_j = -\underline{\nabla}_j\underline{\nabla}_i$ for $i \neq j$

Questions • What is $\mathbb{C}\langle \underline{x}_1, \ldots, \underline{x}_n \rangle^G$?

— i.e., can the above class of groups be characterised by polynomiality of the invariants?

Example $\mathbb{C}\langle \underline{x}_1, \ldots, \underline{x}_n \rangle^{B_n^+}$ is *polynomial* and is generated by

- $\underline{x}_1^{2k} + \cdots + \underline{x}_n^{2k}, \qquad k = 1, 2, \ldots, n-1;$
- $\underline{x}_1 \underline{x}_2 \ldots \underline{x}_n,$

That is, B_n^+ (not a reflection group in the usual sense) has polynomial “anticommutative invariants” and has exponents $2, 4, \ldots, 2(n-1), n$.

NB: the product of the exponents is precisely $|B_n^+|$.

Kirkman, Kuzmanovich, Zhang (2009) proved [independently of B.-B.]:

$S_{\mathbf{q}}(V)^G$ is q'-polynomial, if and only if G is one of the above B.-B. groups.

(*This settles the C-S-T theorem for* $S_{\mathbf{q}}(V)^G$ — **Problem 1** is now solved.)

- The algebra of q-commuting variables $x_1, \ldots, x_n$ (the quantum hyperplane):

if $q \neq -1$, need to consider finite-dimensional quotients of Manin's quantum group $GL_q(n, \mathbb{C})$;

"Dunkl operators" will be a deformation of the Wess-Zumino braided differential calculus.

Thank you.